FINE GROUP GRADINGS OF THE REAL FORMS OF $sl(4,\mathbb{C})$, $sp(4,\mathbb{C})$, AND $o(4,\mathbb{C})$

JIŘÍ PATERA, EDITA PELANTOVÁ, AND MILENA SVOBODOVÁ

Abstract. We present an explicit description of the 'fine group gradings' (i.e. group gradings which cannot be further refined) of the real forms of the semisimple Lie algebras $sl(4,\mathbb{C})$, $sp(4,\mathbb{C})$, and $o(4,\mathbb{C})$. All together 12 real Lie algebras are considered, and the total of 44 of their fine group gradings are listed.

The inclusions $sl(4,\mathbb{C}) \supset sp(4,\mathbb{C}) \supset o(4,\mathbb{C})$ are an important tool in our presentation. Systematic use is made of the faithful representations of the three Lie algebras by 4×4 matrices.

1. Introduction

Wide diversity of applications of low dimensional semisimple Lie algebras, real or complex, is witnessed by the innumerable papers found in the literature, where such Lie algebras play a role. In the article we consider the real forms of the three semisimple Lie algebras which are faithfully represented by 4×4 matrices. More precisely, we have the Lie algebras and the inclusions among them:

$$sl(4,\mathbb{C}) \supset sp(4,\mathbb{C}) \supset o(4,\mathbb{C}).$$

Here $sl(4,\mathbb{C})$ is the Lie algebra of all traceless matrices $\mathbb{C}^{4\times 4}$. Then $sp(4,\mathbb{C})$ is the Lie algebra of all (symplectic) transformations which preserve a skew-symmetric bilinear form in the 4-space $\mathbb{C}^4$. Finally, $o(4,\mathbb{C})$ is the Lie algebra of all (orthogonal) transformations which preserve a symmetric bilinear form in $\mathbb{C}^4$. The second inclusion is somewhat misleading and therefore deserves a comment. Due to the fact that $o(4,\mathbb{C}) \simeq sl(2,\mathbb{C}) \times sl(2,\mathbb{C})$ is not simple, one has a choice considering $o(4,\mathbb{C})$ as linear transformations in $\mathbb{C}^4$. One can introduce symmetric or skew-symmetric bilinear form in $\mathbb{C}^4$ invariant under the matrices of $o(4,\mathbb{C})$. In the symmetric case, $o(4,\mathbb{C})$ contains all such transformations, while all symplectic transformations are in $sp(4,\mathbb{C})$ and only some of them in $o(4,\mathbb{C})$.

The list of the real forms considered in this paper is the following:

$$sl(4,\mathbb{C}) \;:\quad sl(4,\mathbb{R}),\ su^*(4),\ su(4,0),\ su(3,1),\ su(2,2) \tag{1}$$

$$sp(4,\mathbb{C}) \;:\quad sp(4,\mathbb{R}),\ usp(4,0),\ usp(2,2) \tag{2}$$

$$o(4,\mathbb{C}) \;:\quad so^*(4),\ so(4,0),\ so(3,1),\ so(2,2) \tag{3}$$

The subject of our paper are the gradings of these real forms. For a general motivation for studying the gradings, we can point out [10, 11, 12, 13, 20, 21]. Applications of the real forms are too numerous to be mentioned here. For example, let us just mention that among the orthogonal realizations of B_2 there are the Lie algebras of the de Sitter groups, that some real forms of the symplectic realization have been intensively applied in the recent years in nuclear physics, and also in quantum optics [18, 19, 1, 3].

A grading carries basic structural information about its Lie algebra. That is particularly true about the fine grading. One type of a grading is obtained by decomposing the Lie algebra into eigensubspaces of mutually commuting automorphisms. This grading has an interesting property: Its grading subspaces can be indexed by elements of an Abelian group. Such gradings are called group gradings. The most famous example of a grading of this type, obtained by means of automorphisms from the MAD-group which is the maximal torus of the corresponding Lie group, is called the root decomposition. It underlies most of the representation theory. As was recently shown in [6], not all gradings can be associated with groups of automorphisms. In this article we only concentrate on group gradings.

Date: February 7, 2008.

The coarsest non-trivial grading is by the group $\mathbb{Z}_2$ of order 2. There is a one-to-one correspondence between real forms of a Lie algebra, say L, and non-equivalent $\mathbb{Z}_2$ groups inside of the group of automorphisms of L.

Motivation of a physicist to study fine gradings may come from rather different sources. Let us mention just two: (i) For physical system which display a symmetry of L, it is interesting to know all MAD-groups, because their eigenvalues are the additive quantum numbers characterizing the state of such system. (ii) A meaningful relation between physical systems that display different symmetry Lie algebras, can be established if the Lie algebras are related by a process of singular deformations ('contractions'). Grading preserving contractions is a powerful method for describing the wealth of possible deformations.

A classification of MAD-groups of classical Lie algebras over $\mathbb{C}$ is found in [7]; similarly one finds in [8] the classification of MAD-groups of their real forms. Practically, however, concise presentation in [7, 8] leaves a laborious task to a reader interested in specific realization of the gradings of particular Lie algebras. References [14, 15, 16] are the departure points of the present work. Gradings of the exceptional Lie algebra g_2 are listed in [4], gradings of f_4 can be found in [5].

2. Preliminaries

For a Lie algebra $(L, [\ ])$ we define a **grading** Γ as a decomposition of the vector space L into a direct sum of subspaces $\{0\} \neq L_k \subseteq L$, $k \in \mathcal{K}$ fulfilling the property $[L_k, L_l] \subseteq L_m$ for some $m \in \mathcal{K}$. It means, in other words, that for each $k, l \in \mathcal{K}$ there exists $m \in \mathcal{K}$ such that $[L_k, L_l] = \{[X_k, X_l] \,|\, X_k \in L_k, X_l \in L_l\} \subseteq L_m$.

We denote such decomposition by $\Gamma : L = \oplus_{k\in\mathcal{K}} L_k$.

We call a grading $\widetilde{\Gamma}$ of L a **refinement** of a grading $\Gamma : L = \oplus_{k\in\mathcal{K}} L_k$ if $\widetilde{\Gamma} : L = \oplus_{k\in\mathcal{K}, i\in\mathcal{I}_k} L_{ki}$ and $L_k = \oplus_{i\in\mathcal{I}_k} L_{ki}$ for each $k \in \mathcal{K}$.

By means of refinements, we can gradually split the gradings of L until it is impossible to make any further non-trivial refinement. Such a grading whose each refinement is equal to itself is called a fine grading. Overall, the set of all gradings of the Lie algebra L forms a structure of a 'tree', with L itself on the top and the fine gradings of L on the bottom.

Having a set of diagonalizable automorphisms that mutually commute, it is possible to split the Lie algebra into subspaces that are mutual eigensubspaces of all the automorphisms from this set, thus obtaining a grading. Obviously, the bigger set of automorphisms we use, the finer grading we receive by this decomposition. Therefore, for studying the gradings of Lie algebras, an important role is played by so-called MAD-groups. The term **MAD-group** of a Lie algebra L is a short for maximal Abelian subgroup of diagonalizable automorphisms (contained in the group $\mathcal{A}ut\, L$ of all automorphisms on the Lie algebra L).

Indices of a grading obtained in this way can be embedded into a group, thus enabling us to introduce the term of a group grading. We say that the grading $\Gamma : L = \oplus_{k\in\mathcal{K}} L_k$ is a **group grading** if the index set $\mathcal{K}$ can be chosen as a subset of a group in such a way that $\{0\} \neq [L_j, L_k]$ implies $[L_j, L_k] \subseteq L_{j\star k}$. Note that for two different grading subspaces L_j and L_k the corresponding group elements j, k must be different. If a refinement of a group gradings is also a group grading, then we call it **group refinement**. There is a relationship between the group labelling the group refinement and the group labelling the original group grading (see [4]). More precisely, the index set of a group grading can be embedded into a universal Abelian group, say G, such that the index set of any coarsening is embedable into an image of G by a group epimorphism.

There is now a big confusion in literature between the terms grading and group grading, caused by an incorrect statement from [17], which says that the index set of each grading of a Lie algebra is embedable into a semigroup. As already mentioned, there exist counterexamples to this statement, some of them provided firstly by [6]. On the basis of that incorrect statement, the following theorem was proved in [17]:

For a simple Lie algebra L over an algebraically closed field, a grading Γ of L is fine if and only if there exists a MAD-group $\mathcal{G} \subset \mathcal{A}ut\, L$ such that Γ is a decomposition of L into simultaneous eigensubspaces of all automorphisms from $\mathcal{G}$.

It has not been shown yet whether this theorem is true, or not.

We have been often referring to this theorem in our previous works ([14, 15, 16]), and concluding that for the Lie algebras studied in there we have found all the fine gradings. In fact, we may use only the following weaker statement about relation between MAD-groups and fine gradings.

Theorem 1. *A group grading Γ of a complex Lie algebra L is fine, if and only if there exists a MAD-group $\mathcal{G} \subset \mathcal{A}ut\, L$ such that Γ is obtained as a decomposition of L into simultaneous eigensubspaces of all automorphisms from $\mathcal{G}$.*

It means that, in the articles ([14, 15, 16]) cited above, we have described all fine group gradings, but we cannot claim that they are at the same time all the fine gradings. On the example of the non-simple algebra $o(4,\mathbb{C})$ we have found a fine grading which is not a group grading (see the case of grading by the MAD-group $\mathcal{G}_5$ in Table 11). If we would try to find the indices of the subspaces in the fine grading in order to obtain a group grading, then two of the grading subspaces would have to be labelled by the same index, thus giving rise to a coarser group grading. Note that this happens for any (non-simple) Lie algebra L in the form $L = L_1 \otimes L_2$, where the grading arises as a composition of the two root gradings of the individual algebras L_1 and L_2. These findings lead us to the following assumption:

Conjecture 2. *On a simple complex Lie algebra, the terms fine grading and fine group grading coincide.*

To summarize, we have a one-to-one correspondence between the fine group gradings and the MAD-groups for complex Lie algebras. And since the MAD-groups were fully classified for all the 'standard' complex Lie algebras (meaning the classes A_n, B_n, C_n, D_n, with the exception of D_4), we can obtain all the fine group gradings for these algebras. However, we cannot conclude that these fine group gradings are at the same time all the fine gradings of the particular algebras, although we expect that it is the case for simple algebras.

For a grading $\Gamma : L = \oplus_{k\in\mathcal{K}} L_k$ of L and an automorphism $g \in \mathcal{A}ut\, L$, the decomposition $\widetilde{\Gamma} : L = \oplus_{k\in\mathcal{K}}(gL_k)$ is also a grading of L; and we call such gradings Γ and $\widetilde{\Gamma}$ equivalent. Further on, when speaking about two 'different' gradings, we mean non-equivalent gradings. Consequently, if Γ is a group grading, then the equivalent grading $\widetilde{\Gamma}$ is again a group grading with the same index set $\mathcal{K}$ lying in the same group.

For real Lie algebras, however, we do not possess any simple tool like MAD-groups. Our aim in this work is to describe fine group gradings of all the real forms of the Lie algebras $sl(4,\mathbb{C})$, $sp(4,\mathbb{C})$, and $o(4,\mathbb{C})$, and relationships between them. For classical simple complex Lie algebras, these relationships are very straightforward, given the fact that $sp(4,\mathbb{C})$ and $o(4,\mathbb{C})$ are subalgebras of $sl(4,\mathbb{C})$. It was proved in [7] that a fine group grading of $sp(n,\mathbb{C})$ or $o(n,\mathbb{C})$, $n \neq 8$, is always formed as a selection of several grading subspaces from a fine group grading of $sl(n,\mathbb{C})$. We will use this method also for finding fine group gradings of real forms of the Lie algebras in question.

Firstly, let us recall previous results ([14, 15, 16]) containing fine group gradings of the complex Lie algebras $sl(4,\mathbb{C})$, $sp(4,\mathbb{C})$, and $o(4,\mathbb{C})$ themselves, because they are the starting point for exploration of fine group gradings of their real forms. Then we shall move on to the core of this work - namely the fine group gradings of the real forms of the three Lie algebras in question. Note that gradings of $sp(4,\mathbb{C})$ are also studied in [2].

3. Fine Group Gradings of the Complex Lie Algebras $sl(4,\mathbb{C})$, $sp(4,\mathbb{C})$, and $o(4,\mathbb{C})$

3.1. **Fine Group Gradings of the Complex Lie Algebra $sl(4,\mathbb{C})$.**

Let us start with the algebra $sl(4,\mathbb{C}) = \{X \in \mathbb{C}^{4\times 4} \mid \mathrm{tr}(X) = \sum_{j=1}^{4} X_{jj} = 0\}$. On $sl(4,\mathbb{C})$, there exist two types of automorphisms:

- inner automorphism: Ad_A for $A \in Gl(4,\mathbb{C})$, where $Ad_A(X) = A^{-1}XA$;
- outer automorphism: Out_C for $C \in Gl(4,\mathbb{C})$, where $Out_C(X) = -(C^{-1}XC)^T$.

The full set of automorphisms on $sl(4,\mathbb{C})$ is then

$$\mathcal{A}ut\, sl(4,\mathbb{C}) = \{Ad_A \mid \det A \neq 0\} \cup \{Out_C \mid \det C \neq 0\}.$$

In our list of MAD-groups, we always characterize each MAD-group by the set G_{Ad} of matrices A for the inner automorphisms, and by the set G_{Out} of matrices C for the outer automorphisms if the MAD-group contains outer automorphisms at all.

In fact, we provide just one element C of G_{Out}, because any other outer automorphism $Out_{\widetilde{C}}$ from the MAD-group is a composition of the one Out_C and of some inner automorphism Ad_A with $A \in G_{Ad}$.

There are eight MAD-groups on $sl(4,\mathbb{C})$ listed in Table 1; two of them with inner automorphisms only, and six of them containing also outer automorphisms. The symbol $\otimes$ used in the Table stands for the tensor product of matrices.[1]

	G_{Ad}	G_{Out}
$\mathcal{G}_1$	$\{A = \mathrm{diag}\,(d_1, d_2, d_3, 1),\ d_j \in \mathbb{C}, d_j \neq 0\}$	$\emptyset$
$\mathcal{G}_2$	$\{A = P^j Q^k,\ j, k = 0, 1, 2, 3,$ $P = \left(\begin{smallmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0 \end{smallmatrix}\right), Q = \mathrm{diag}\,(1, i, -1, -i)\}$	$\emptyset$
$\mathcal{G}_3$	$\{A = \mathrm{diag}\,(\varepsilon_1, \varepsilon_2, \varepsilon_3, 1),\ \varepsilon_i = \pm 1\}$	$C = I_4$
$\mathcal{G}_4$	$\{A = \mathrm{diag}\,(1, \varepsilon, \alpha, \alpha^{-1}),\ \varepsilon = \pm 1, \alpha \in \mathbb{C}, \alpha \neq 0\}$	$C = \left(\begin{smallmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{smallmatrix}\right)$
$\mathcal{G}_5$	$\{A = \mathrm{diag}\,(\alpha, \alpha^{-1}, \beta, \beta^{-1}),\ \alpha, \beta \in \mathbb{C}, \alpha, \beta \neq 0\}$	$C = \left(\begin{smallmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{smallmatrix}\right)$
$\mathcal{G}_6$	$\{A = \sigma_j \otimes \mathrm{diag}\,(\alpha, \alpha^{-1}),\ \alpha \in \mathbb{C}, \alpha \neq 0, j = 0, 1, 2, 3\}$	$C = \left(\begin{smallmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{smallmatrix}\right)$
$\mathcal{G}_7$	$\{A = \sigma_j \otimes \sigma_k,\ j, k = 0, 1, 2, 3\}$	$C = I_4$
$\mathcal{G}_8$	$\{A \in (G_0 \otimes I_2) \cup (RG_0 \otimes \sigma_3) \cup (G_0 \otimes \sigma_1) \cup (RG_0 \otimes \sigma_2),$ $G_0 = \{\left(\begin{smallmatrix} 1&0\\ 0&1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&0\\ 0&-1 \end{smallmatrix}\right)\}, R = \left(\begin{smallmatrix} 1&0\\ 0&i \end{smallmatrix}\right)\}$	$C = \left(\begin{smallmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{smallmatrix}\right)$

TABLE 1. The eight non-conjugate MAD-groups on $sl(4,\mathbb{C})$. The symbols σ_j stand for the 2×2 real Pauli matrices .

Non-conjugate MAD-groups generate non-equivalent fine group gradings of the Lie algebra. For $sl(4,\mathbb{C})$, the Theorem 1 implies that the eight fine group gradings generated by the MAD-groups $\mathcal{G}_1, \dots, \mathcal{G}_8$ form the full set of non-equivalent fine group gradings on $sl(4,\mathbb{C})$.

In the following tables, we provide an explicit summary of all these eight fine group gradings, together with the universal Abelian groups for their index sets. We will use the notation of

- $\underline{L_k \subset sl(4,\mathbb{C})}$: grading subspace L_k of a grading Γ of $sl(4,\mathbb{C})$ described as a complex span of the relevant basis vectors X_i, namely $L_k = \mathbb{C}\cdot X_{i_1} + \ldots + \mathbb{C}\cdot X_{i_k} = \mathrm{span}^{\mathbb{C}}(X_{i_1}, \dots, X_{i_k})$;
- $\underline{E_{jk} \in \mathbb{C}^{4\times 4}}$: matrices containing just one non-zero element and fifteen zeros, namely $(E_{jk})_{lm} = \delta_{jl}\delta_{km}$, where δ_{hi} is the Kronecker symbol.

$L_1 : X_1 = E_{23}$	$L_2 : X_2 = E_{12}$	$L_3 : X_3 = E_{13}$	$L_4 : X_4 = E_{34}$
$L_5 : X_5 = E_{24}$	$L_6 : X_6 = E_{41}$	$L_7 : X_7 = E_{14}$	$L_8 : X_8 = E_{42}$
$L_9 : X_9 = E_{43}$	$L_{10} : X_{10} = E_{31}$	$L_{11} : X_{11} = E_{21}$	$L_{12} : X_{12} = E_{32}$
$L_{13}:\ X_{13} = E_{11} - E_{22} \quad X_{14} = E_{22} - E_{33} \quad X_{15} = E_{33} - E_{44}$			

TABLE 2. Fine group grading $\Gamma_1 : sl(4,\mathbb{C}) = \oplus_{k=1}^{13} L_k$ generated by MAD-group $\mathcal{G}_1$, known as the 'root grading'. There are twelve one-dimensional subspaces $L_k = \mathbb{C}\cdot X_k$, $k = 1, \dots, 12$, and one three-dimensional grading subspace $L_{13} = \mathbb{C}\cdot X_{13} + \mathbb{C}\cdot X_{14} + \mathbb{C}\cdot X_{15}$. The universal Abelian group of this fine group grading is $\mathbb{Z}^3$.

[1] If $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{m\times m}$, then the tensor product $A\otimes B \in \mathbb{C}^{nm\times nm}$ is defined by $(A\otimes B)_{IJ} = A_{i_1 i_2} B_{j_1 j_2}$, where $i_1, i_2 \in \{0, 1, \dots, n-1\}$, $j_1, j_2 \in \{0, 1, \dots, m-1\}$, $I, J \in \{0, 1, \dots, mn-1\}$ and $I = i_1 m + j_1$, $J = i_2 m + j_2$.

$L_1 : X_1 = P^0Q^3$	$L_2 : X_2 = P^0Q^2$	$L_3 : X_3 = P^0Q^1$	$L_4 : X_4 = P^1Q^0$	$L_5 : X_5 = P^1Q^3$
$L_6 : X_6 = P^1Q^2$	$L_7 : X_7 = P^1Q^1$	$L_8 : X_8 = P^2Q^0$	$L_9 : X_9 = P^2Q^3$	$L_{10} : X_{10} = P^2Q^2$
$L_{11} : X_{11} = P^2Q^1$	$L_{12} : X_{12} = P^3Q^0$	$L_{13} : X_{13} = P^3Q^3$	$L_{14} : X_{14} = P^3Q^2$	$L_{15} : X_{15} = P^3Q^1$

TABLE 3. Fine group grading $\Gamma_2 : sl(4,\mathbb{C}) = \oplus_{k=1}^{15} L_k$ generated by MAD-group $\mathcal{G}_2$, known as the 'Pauli grading'. All the grading subspaces are one-dimensional: $L_k = \mathbb{C} \cdot X_k$, $k = 1, \ldots, 15$; the basis matrices X_k are expressed by means of the 4×4 Pauli matrices P, Q defined in Table 1. The universal Abelian group of this fine group grading is $\mathbb{Z}_4^2$.

$L_1 : X_1 = E_{14} + E_{41}$	$L_2 : X_2 = E_{14} - E_{41}$	$L_3 : X_3 = E_{24} + E_{42}$
$L_4 : X_4 = E_{24} - E_{42}$	$L_5 : X_5 = E_{34} + E_{43}$	$L_6 : X_6 = E_{34} - E_{43}$
$L_7 : X_7 = E_{13} + E_{31}$	$L_8 : X_8 = E_{13} - E_{31}$	$L_9 : X_9 = E_{23} + E_{32}$
$L_{10} : X_{10} = E_{23} - E_{32}$	$L_{11} : X_{11} = E_{12} + E_{21}$	$L_{12} : X_{12} = E_{12} - E_{21}$
L_{13}: $X_{13} = E_{11} - E_{22}$ $X_{14} = E_{22} - E_{33}$ $X_{15} = E_{33} - E_{44}$		

TABLE 4. Fine group grading $\Gamma_3 : sl(4,\mathbb{C}) = \oplus_{k=1}^{13} L_k$ generated by MAD-group $\mathcal{G}_3$. There are twelve one-dimensional subspaces $L_k = \mathbb{C} \cdot X_k$, $k = 1, \ldots, 12$, and one three-dimensional grading subspace $L_{13} = \mathbb{C} \cdot X_{13} + \mathbb{C} \cdot X_{14} + \mathbb{C} \cdot X_{15}$. The universal Abelian group of this fine group grading is $\mathbb{Z}_2^4$.

$L_1 : X_1 = E_{13} - E_{41}$	$L_2 : X_2 = E_{13} + E_{41}$	$L_3 : X_3 = E_{14} - E_{31}$
$L_4 : X_4 = E_{14} + E_{31}$	$L_5 : X_5 = E_{24} - E_{32}$	$L_6 : X_6 = E_{24} + E_{32}$
$L_7 : X_7 = E_{23} - E_{42}$	$L_8 : X_8 = E_{23} + E_{42}$	$L_9 : X_9 = E_{12} - E_{21}$
$L_{10} : X_{10} = E_{12} + E_{21}$	$L_{11} : X_{11} = E_{34}$	$L_{12} : X_{12} = E_{43}$
$L_{13} : X_{13} = E_{33} - E_{44}$	L_{14}: $X_{14} = E_{11} - E_{22}$ $X_{15} = E_{11} + E_{22} - E_{33} - E_{44}$	

TABLE 5. Fine group grading $\Gamma_4 : sl(4,\mathbb{C}) = \oplus_{k=1}^{14} L_k$ generated by MAD-group $\mathcal{G}_4$. Thirteen grading subspaces are one-dimensional: $L_k = \mathbb{C} \cdot X_k$, $k = 1, \ldots, 13$, one is two-dimensional: $L_{14} = \mathbb{C} \cdot X_{14} + \mathbb{C} \cdot X_{15}$. The universal Abelian group of this fine group grading is $\mathbb{Z} \times \mathbb{Z}_2^2$.

$L_1 : X_1 = E_{23} - E_{41}$	$L_6 : X_6 = E_{13} + E_{42}$	$L_{11} : X_{11} = E_{34}$
$L_2 : X_2 = E_{23} + E_{41}$	$L_7 : X_7 = E_{14} - E_{32}$	$L_{12} : X_{12} = E_{43}$
$L_3 : X_3 = E_{24} - E_{31}$	$L_8 : X_8 = E_{14} + E_{32}$	$L_{13} : X_{13} = E_{11} + E_{22} - E_{33} - E_{44}$
$L_4 : X_4 = E_{24} + E_{31}$	$L_9 : X_9 = E_{12}$	$L_{14} : X_{14} = E_{11} - E_{22} + E_{33} - E_{44}$
$L_5 : X_5 = E_{13} - E_{42}$	$L_{10} : X_{10} = E_{21}$	$X_{15} = E_{11} - E_{22} - E_{33} + E_{44}$

TABLE 6. Fine group grading $\Gamma_5 : sl(4,\mathbb{C}) = \oplus_{k=1}^{14} L_k$ generated by MAD-group $\mathcal{G}_5$. There are thirteen one-dimensional grading subspaces: $L_k = \mathbb{C} \cdot X_k$, $k = 1, \ldots, 13$, and one two-dimensional grading subspace: $L_{14} = \mathbb{C} \cdot X_{14} + \mathbb{C} \cdot X_{15}$. The universal Abelian group of this fine group grading is $\mathbb{Z}^2 \times \mathbb{Z}_2$.

3.2. **Fine Group Gradings of the Complex Lie Algebras $sp(4,\mathbb{C})$ and $o(4,\mathbb{C})$.** Let us now proceed with the Lie algebras $sp(4,\mathbb{C})$ and $o(4,\mathbb{C})$. Both these algebras are subalgebras of $sl(4,\mathbb{C})$, and they have various representations determined by non-singular matrices $K \in \mathbb{C}^{4\times 4}$, with $K = -K^T$ for $sp_K(4,\mathbb{C})$, and $K = K^T$ for $o_K(4,\mathbb{C})$:

$$sp_K(4,\mathbb{C}) = \{X \in \mathbb{C}^{4\times 4} \,|\, XK = -KX^T\}, \ \det K \neq 0, \ K = -K^T,$$

$$o_K(4,\mathbb{C}) = \{X \in \mathbb{C}^{4\times 4} \,|\, XK = -KX^T\}, \ \det K \neq 0, \ K = K^T.$$

We have two options how to find the fine group gradings of these two Lie algebras:

$L_1 : X_1 = E_{11} - E_{22} + E_{33} - E_{44}$	$L_6 : X_6 = E_{13} + E_{24} + E_{31} + E_{42}$	$L_{11} : X_{11} = E_{14} + E_{32}$
$L_2 : X_2 = E_{11} - E_{22} - E_{33} + E_{44}$	$L_7 : X_7 = E_{13} - E_{24} - E_{31} + E_{42}$	$L_{12} : X_{12} = E_{23} - E_{41}$
$L_3 : X_3 = E_{11} + E_{22} - E_{33} - E_{44}$	$L_8 : X_8 = E_{21} + E_{43}$	$L_{13} : X_{13} = E_{23} + E_{41}$
$L_4 : X_4 = E_{13} - E_{24} + E_{31} - E_{42}$	$L_9 : X_9 = E_{21} - E_{43}$	$L_{14} : X_{14} = E_{12} + E_{34}$
$L_5 : X_5 = E_{13} + E_{24} - E_{31} - E_{42}$	$L_{10} : X_{10} = E_{14} - E_{32}$	$L_{15} : X_{15} = E_{12} - E_{34}$

Table 7. Fine group grading $\Gamma_6 : sl(4,\mathbb{C}) = \oplus_{k=1}^{15} L_k$ generated by MAD-group $\mathcal{G}_6$. All of the grading subspaces are one-dimensional: $L_k = \mathbb{C} \cdot X_k$, $k = 1, \ldots, 15$. The universal Abelian group of this fine group grading is $\mathbb{Z} \times \mathbb{Z}_2^3$.

$L_1 : X_1 = E_{14} + E_{23} + E_{32} + E_{41}$	$L_9 : X_9 = E_{12} + E_{21} + E_{34} + E_{43}$
$L_2 : X_2 = E_{14} - E_{23} - E_{32} + E_{41}$	$L_{10} : X_{10} = E_{12} - E_{21} - E_{34} + E_{43}$
$L_3 : X_3 = E_{14} - E_{23} + E_{32} - E_{41}$	$L_{11} : X_{11} = E_{12} - E_{21} + E_{34} - E_{43}$
$L_4 : X_4 = E_{14} + E_{23} - E_{32} - E_{41}$	$L_{12} : X_{12} = E_{12} + E_{21} - E_{34} - E_{43}$
$L_5 : X_5 = E_{13} + E_{24} + E_{31} + E_{42}$	$L_{13} : X_{13} = E_{11} - E_{22} - E_{33} + E_{44}$
$L_6 : X_6 = E_{13} - E_{24} - E_{31} + E_{42}$	$L_{14} : X_{14} = E_{11} - E_{22} + E_{33} - E_{44}$
$L_7 : X_7 = E_{13} - E_{24} + E_{31} - E_{42}$	$L_{15} : X_{15} = E_{11} + E_{22} - E_{33} - E_{44}$
$L_8 : X_8 = E_{13} + E_{24} - E_{31} - E_{42}$	

Table 8. Fine group grading $\Gamma_7 : sl(4,\mathbb{C}) = \oplus_{k=1}^{15} L_k$ generated by MAD-group $\mathcal{G}_7$. All of the grading subspaces are one-dimensional: $L_k = \mathbb{C} \cdot X_k$, $k = 1, \ldots, 15$. The universal Abelian group of this fine group grading is $\mathbb{Z}_2^5$.

$L_1 : X_1 = E_{13} + E_{24} + E_{32} + E_{41}$	$L_6 : X_6 = E_{14} - E_{23} - E_{31} + E_{42}$	$L_{11} : X_{11} = E_{33} - E_{44}$
$L_2 : X_2 = E_{13} - E_{24} - E_{32} + E_{41}$	$L_7 : X_7 = E_{14} - E_{23} + E_{31} - E_{42}$	$L_{12} : X_{12} = E_{12} - E_{21}$
$L_3 : X_3 = E_{13} - E_{24} + E_{32} - E_{41}$	$L_8 : X_8 = E_{14} + E_{23} - E_{31} - E_{42}$	$L_{13} : X_{13} = E_{34} - E_{43}$
$L_4 : X_4 = E_{13} + E_{24} - E_{32} - E_{41}$	$L_9 : X_9 = E_{11} + E_{22} - E_{33} - E_{44}$	$L_{14} : X_{14} = E_{12} + E_{21}$
$L_5 : X_5 = E_{14} + E_{23} + E_{31} + E_{42}$	$L_{10} : X_{10} = E_{11} - E_{22}$	$X_{15} = E_{34} + E_{43}$

Table 9. Fine group grading $\Gamma_8 : sl(4,\mathbb{C}) = \oplus_{k=1}^{14} L_k$ generated by MAD-group $\mathcal{G}_8$. There are thirteen one-dimensional grading subspaces: $L_k = \mathbb{C} \cdot X_k$, $k = 1, \ldots, 13$, and one two-dimensional grading subspace: $L_{14} = \mathbb{C} \cdot X_{14} + \mathbb{C} \cdot X_{15}$. The universal Abelian group of this fine group grading is $\mathbb{Z}_4 \times \mathbb{Z}_2^2$.

- 'MAD-group' method: Being Lie algebras themselves, $sp_K(4,\mathbb{C})$ and $o_K(4,\mathbb{C})$ have MAD-groups in their automorphism groups. We can use these MAD-groups for splitting the algebras into simultaneous eigensubspaces, thus obtaining fine group gradings.
- 'Displayed' method: As each fine group grading of $sp_K(4,\mathbb{C})$ and $o_K(4,\mathbb{C})$ can be extended to a fine group grading of $sl(4,\mathbb{C})$, we have the possibility to find the fine group gradings of $sp_K(4,\mathbb{C})$ and $o_K(4,\mathbb{C})$ by means of the 'Displayed' method. We take the fine group gradings of $sl(4,\mathbb{C})$ and select some of the grading subspaces, so that they make up the respective subalgebra, already being in the form of a grading.

The success of the 'Displayed' method is ensured by the following statement (proved in [7]):

Theorem 3. *Let Γ be a fine group grading of $sl(n,\mathbb{C})$ generated by MAD-group $\mathcal{G}$. This grading Γ displays the subalgebra $o_K(n,\mathbb{C})$ for $n \neq 8$, or $sp_K(n,\mathbb{C})$, for matrices verifying $K = K^T$, or $K = -K^T$ respectively, if and only if $\mathcal{G}$ contains the automorphism Out_K.*

Practically, the usage of Theorem 3 reduces into choosing those grading subspaces from the grading Γ which are eigensubspaces of this automorphism Out_K corresponding to eigenvalue $+1$. Indeed, this property is equivalent to being an element of $sp_K(n,\mathbb{C})$, $o_K(n,\mathbb{C})$ respectively:

$$X = Out_K X = -(K^{-1}XK)^T \qquad \Leftrightarrow \qquad XK = -KX^T.$$

The two methods ('MAD-group' and 'Displayed') lead in principle to the same result. The advantage of the 'Displayed' method is that the grading subspaces of $sp_K(4,\mathbb{C})$ coincide with

chosen grading subspaces of $sl(4,\mathbb{C})$, the same goes for $o_K(4,\mathbb{C})$. This comfort is paid a price for, though, because the fine gradings are then expressed in various representations of $o_K(4,\mathbb{C})$. (For $sp_K(4,\mathbb{C})$, we luckily happen to obtain all the fine group gradings in the same representation even by the 'Displayed' method.)

We obtain all the three fine group gradings of $sp_K(4,\mathbb{C})$ and all the six fine group gradings of $o_K(4,\mathbb{C})$. All these fine group gradings found in this way on $sp_K(4,\mathbb{C})$ and $o_K(4,\mathbb{C})$ (up to one exception) are at the same time fine gradings, as is implied by the dimensions of the grading subspaces. The exceptional case is the fine group grading of $o_K(4,\mathbb{C})$ displayed by the fine group grading Γ_5 of $sl(4,\mathbb{C})$ generated by the MAD-group $\mathcal{G}_5$ (see Table 11). It is not a fine (non-group) grading of $o_K(4,\mathbb{C})$, because it can be further refined, but this refinement is not a group grading any more.

In Tables 10 and 11, we list the fine group gradings of $sp_K(4,\mathbb{C})$ and $o_K(4,\mathbb{C})$ in the form as provided by the 'Displayed' method, because this notation will be useful later on for description of fine group gradings of their real forms. We only need to keep in mind that the fine group gradings of $sl(4,\mathbb{C})$ display the fine group gradings of $o_K(4,\mathbb{C})$ in several different representations - i.e. defined by different matrices $K = K^T$.

MAD-group on $sl(4,\mathbb{C})$	defining matrix $K = -K^T$	grading subspaces of $sl(4,\mathbb{C})$ making up the respective fine group grading of $sp_K(4,\mathbb{C})$
$\mathcal{G}_5$		$L_{14} \oplus L_2 \oplus L_{11} \oplus L_3 \oplus L_{10} \oplus L_9 \oplus L_5 \oplus L_{12} \oplus L_8$
$\mathcal{G}_6$	$K_0 = J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$L_1 \oplus L_2 \oplus L_8 \oplus L_9 \oplus L_{11} \oplus L_4 \oplus L_5 \oplus L_{13} \oplus L_{14} \oplus L_{15}$
$\mathcal{G}_7$		$L_{13} \oplus L_{14} \oplus L_1 \oplus L_3 \oplus L_8 \oplus L_7 \oplus L_{12} \oplus L_{11} \oplus L_{10} \oplus L_9$

TABLE 10. Fine group gradings of $sp_K(4,\mathbb{C})$ as displayed by fine group gradings of $sl(4,\mathbb{C})$. In all the three cases, the algebra $sp_K(4,\mathbb{C})$ happens to be displayed in the same representation.

MAD-group on $sl(4,\mathbb{C})$	defining matrix $K = K^T$	grading subspaces of $sl(4,\mathbb{C})$ making up the respective fine group grading of $o_K(4,\mathbb{C})$
$\mathcal{G}_3$	$K_1 = I = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}$	$L_2 \oplus L_4 \oplus L_6 \oplus L_8 \oplus L_{10} \oplus L_{12}$
$\mathcal{G}_4$	$K_2 = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix}$	$L_{13} \oplus L_1 \oplus L_5 \oplus L_9 \oplus L_7 \oplus L_3$
$\mathcal{G}_5$	$\rightarrow$ fine group grading displayed by the fine group grading Γ_5 of $sl(4,\mathbb{C})$, but not a fine grading: $L_{14} \oplus L_1 \oplus L_3 \oplus L_5 \oplus L_7$ $\rightarrow$ we split L_{14} into $L_{14}^1 = \mathbb{C} \cdot X_{14}$, $L_{14}^2 = \mathbb{C} \cdot X_{15}$ $\rightarrow$ we get a fine (non-group) grading:	
	$K_3 = \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix}$	$L_{14}^1 \oplus L_{14}^2 \oplus L_1 \oplus L_3 \oplus L_5 \oplus L_7$
$\mathcal{G}_6$	$K_3 = \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix}$	$L_1 \oplus L_2 \oplus L_{10} \oplus L_4 \oplus L_5 \oplus L_{12}$
$\mathcal{G}_7$	$K_1 = I = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}$	$L_3 \oplus L_4 \oplus L_8 \oplus L_6 \oplus L_{11} \oplus L_{10}$
$\mathcal{G}_8$	$K_2 = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix}$	$L_{11} \oplus L_4 \oplus L_3 \oplus L_{12} \oplus L_8 \oplus L_6$

TABLE 11. Fine group gradings of $o_K(4,\mathbb{C})$ as displayed by the fine group gradings of $sl(4,\mathbb{C})$, but in various representations given by different symmetric matrices K. With the exception of $\mathcal{G}_5$, we directly get fine gradings of $o_K(4,\mathbb{C})$; for $\mathcal{G}_5$, we need to split the two-dimensional grading subspace into two, and only then the grading becomes fine.

4. Fine Group Gradings of the Real Forms of $sl(4,\mathbb{C})$, $sp(4,\mathbb{C})$, and $o(4,\mathbb{C})$

A real Lie subalgebra $\widetilde{L}$ of a complex Lie algebra L is called a **real form** of L if each element $x \in L$ is uniquely representable in the form $x = u + iv$, where $u, v \in \widetilde{L}$. Not every complex Lie algebra has real forms. On the other hand, a given complex Lie algebra may, in general, have several non-isomorphic real forms.

Two real forms $\widetilde{L}_1$ and $\widetilde{L}_2$ of a Lie algebra L are said to be **isomorphic** if $g(\widetilde{L}_1) = \widetilde{L}_2$ for some automorphism $g \in \mathcal{A}ut\ L$.

For the classical complex Lie algebras, we have the following description of their real forms (see [9]):

Every real form of the complex Lie algebra L can be expressed as $L_{\mathcal{J}} = \{X \in L \,|\, \mathcal{J}(X) = X\}$, where $\mathcal{J}$ is an involutive antiautomorphism on L. Let us recall the meaning of the term 'involutive antiautomorphism':

- involutive: $\mathcal{J}^2 = Id$;
- antiautomorphism: $\mathcal{J}(\alpha X + Y) = \overline{\alpha}\mathcal{J}(X) + \mathcal{J}(Y)$ for each $X, Y \in L$, $\alpha \in \mathbb{C}$.

This definition of the real forms, together with full classification of the system of involutive antiautomorphisms, provide a practical tool for working with real forms of the complex Lie algebras we are interested in, namely $sl(4,\mathbb{C})$, $sp(4,\mathbb{C})$, and $o(4,\mathbb{C})$.

4.1. Fine Group Gradings of the Real Forms of $sl(4,\mathbb{C})$.

Let us write down the classification of all the real forms of $sl(4,\mathbb{C})$. This complex Lie algebra has five non-isomorphic real forms, and we list them in Table 12 together with the respective involutive antiautomorphisms that define them via $L_{\mathcal{J}} = \{X \in L \,|\, \mathcal{J}(X) = X\}$.

We introduce a special notation for one involutive antiautomorphism on $sl(4,\mathbb{C})$ by $\mathcal{J}_0(X) = \overline{X}$. This mapping $\mathcal{J}_0$ plays an important role, because any involutive antiautomorphism on $sl(4,\mathbb{C})$ can be made up as a composition of $\mathcal{J}_0$ and some automorphism $h \in \mathcal{A}ut\, sl(4,\mathbb{C})$.

a)	$\mathcal{J} = \mathcal{J}_0 Ad_F$ $F\overline{F} = I$ (circular)	$sl(4,\mathbb{R})$	$=L_{\mathcal{J}} = L_{\mathcal{J}_0 Ad_F} =$ $=\{X \in sl(4,\mathbb{C}) \,\|\, \mathcal{J}_0 Ad_F(X) = X\} =$ $=\{X \in sl(4,\mathbb{C}) \,\|\, XF = F\overline{X}\}$
b)	$\mathcal{J} = \mathcal{J}_0 Ad_F$ $F\overline{F} = -I$ (anticircular)	$su^*(4)$	$=L_{\mathcal{J}} = L_{\mathcal{J}_0 Ad_F} =$ $=\{X \in sl(4,\mathbb{C}) \,\|\, \mathcal{J}_0 Ad_F(X) = X\} =$ $=\{X \in sl(4,\mathbb{C}) \,\|\, XF = F\overline{X}\}$
c)	$\mathcal{J} = \mathcal{J}_0 Out_E$ $E = E^+$ (hermitian) $\mathrm{sgn}(E) = (4-r, 0, r)$	$su(4-r, r)$	$=L_{\mathcal{J}} = L_{\mathcal{J}_0 Out_E} =$ $=\{X \in sl(4,\mathbb{C}) \,\|\, \mathcal{J}_0 Out_E(X) = X\} =$ $=\{X \in sl(4,\mathbb{C}) \,\|\, XE = -EX^+\}$
	$r = 0, 1, 2 \to su(4,0),\ su(3,1),\ su(2,2)$		

Table 12. The five non-isomorphic real forms of $sl(4,\mathbb{C})$ and their defining involutive antiautomorphisms. There is just one real form of type a), another one of type b), and three real forms of type c). The notation $\mathrm{sgn}(E)$ stands for a triplet of non-negative integers, where the first one is the number of positive elements in the spectrum of E, second is the number of zeros in the spectrum, and third is the number of negative elements in the spectrum.

Now we get to the point of searching for the fine group gradings of these real forms. One straightforward idea how this could be done is to use the same method as for the complex algebra - namely take the MAD-groups of the real form and split the real form into simultaneous eigensubspaces of all the automorphisms in the MAD-group. Let us denote this process again as the 'MAD-group' method. For this purpose, let us look at the overview of the MAD-groups on the real forms of $sl(4,\mathbb{C})$.

For a group $\mathcal{H}$ of automorphisms on a complex Lie algebra L, we denote by $\mathcal{H}^{\mathbb{R}} = \{h \in \mathcal{H} \,|\, \sigma(h) \subset \mathbb{R}\}$ the so-called **real part** of the group $\mathcal{H}$ - i.e. all automorphisms h from $\mathcal{H}$ with real spectrum $\sigma(h)$.

Let $\mathcal{G} \subset \mathcal{A}ut\ L$ be a MAD-group on the complex Lie algebra L, and let $\mathcal{G}^{\mathbb{R}}$ be its real part. We say that the set $\mathcal{G}^{\mathbb{R}}$ is maximal if there exists no such MAD-group $\widetilde{\mathcal{G}}$ on L that $\mathcal{G}^{\mathbb{R}}$ is conjugate to some proper subgroup of $\widetilde{\mathcal{G}}^{\mathbb{R}}$.

It is proved (in [8]) that each MAD-group $\mathcal{F}$ on a real form of a classical simple Lie algebra L is equal to the maximal real part $\mathcal{G}^{\mathbb{R}}$ of some MAD-group $\mathcal{G}$ on L. (Obviously, $\mathcal{F}$ is a subgroup of $\mathcal{G}^{\mathbb{R}}$ for some MAD-group $\mathcal{G}$ on L, because all automorphisms $g \in \mathcal{F}$ can be uniquely extended from $L_{\mathcal{J}}$ to L by $g^{\mathbb{C}}(X+iY) = g(X) + ig(Y)$ - such extensions are diagonalizable, their spectrum is real, and they mutually commute.)

So, we start with the eight MAD-groups on $sl(4,\mathbb{C})$, and take their real parts. Not all of the real parts are maximal:

- $\mathcal{G}_2^{\mathbb{R}}$ is not maximal, because it is equal to a proper subgroup of $\mathcal{G}_7^{\mathbb{R}}$: $\mathcal{G}_2^{\mathbb{R}} \subsetneqq \mathcal{G}_7^{\mathbb{R}}$;
- $\mathcal{G}_8^{\mathbb{R}}$ is not maximal, because it is conjugate to a proper subgroup of $\mathcal{G}_7^{\mathbb{R}}$:
  $\widetilde{\mathcal{G}}_8^{\mathbb{R}} = f\mathcal{G}_8^{\mathbb{R}} f^{-1} \subsetneqq \mathcal{G}_7^{\mathbb{R}}$, with $f = Ad_S$, $S = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \oplus \begin{pmatrix}\frac{1+i}{2} & \frac{1-i}{2}\\ \frac{1-i}{2} & \frac{1+i}{2}\end{pmatrix}$.

Therefore, the real parts of $\mathcal{G}_2^{\mathbb{R}}$ and $\mathcal{G}_8^{\mathbb{R}}$ are not MAD-groups on any of the five real forms of $sl(4,\mathbb{C})$.

The remaining complex MAD-groups $\mathcal{G}_1$, $\mathcal{G}_3$, $\mathcal{G}_4$, $\mathcal{G}_5$, $\mathcal{G}_6$, $\mathcal{G}_7$ do have maximal real parts $\mathcal{G}_1^{\mathbb{R}}$, $\mathcal{G}_3^{\mathbb{R}}$, $\mathcal{G}_4^{\mathbb{R}}$, $\mathcal{G}_5^{\mathbb{R}}$, $\mathcal{G}_6^{\mathbb{R}}$, $\mathcal{G}_7^{\mathbb{R}}$. See the explicit overview of all the real parts $\mathcal{G}_j^{\mathbb{R}}$, $j = 1,\dots,8$ in Table 13.

	$G_{Ad}^{\mathbb{R}}$	$G_{Out}^{\mathbb{R}}$
$\mathcal{G}_1^{\mathbb{R}}$	$\{A = \mathrm{diag}(d_1, d_2, d_3, 1),\ d_j \in \mathbb{R}, d_j \neq 0\}$	$\emptyset$
$\mathcal{G}_3^{\mathbb{R}} = \mathcal{G}_3$	$\{A = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4),\ \varepsilon_i = \pm 1\}$	$C = I_4$
$\mathcal{G}_4^{\mathbb{R}}$	$\{A = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \alpha, \alpha^{-1}),\ \varepsilon_i = \pm 1, \alpha \in \mathbb{R}, \alpha \neq 0\}$	$C = \left(\begin{smallmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{smallmatrix}\right)$
$\mathcal{G}_5^{\mathbb{R}}$	$\{A = \mathrm{diag}(\alpha, \alpha^{-1}, \beta, \beta^{-1}),$ $(\alpha, \beta \in \mathbb{R}) \vee (\alpha, \beta \in i\mathbb{R}), \alpha, \beta \neq 0\}$	$C = \left(\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{smallmatrix}\right)$
$\mathcal{G}_6^{\mathbb{R}}$	$\{A = \sigma_j \otimes \mathrm{diag}(\alpha, \alpha^{-1}), A = \sigma_j \otimes \mathrm{diag}(\alpha, -\alpha^{-1}),$ $\alpha \in \mathbb{R}, \alpha \neq 0, j = 0,1,2,3\}$	$C = \left(\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{smallmatrix}\right)$
$\mathcal{G}_7^{\mathbb{R}} = \mathcal{G}_7$	$\{A = \sigma_j \otimes \sigma_k,\ j,k = 0,1,2,3\}$	$C = I_4$
$\mathcal{G}_2^{\mathbb{R}} \subsetneqq \mathcal{G}_7^{\mathbb{R}}$	$\{A \in \{\sigma_0 \otimes \sigma_0, \sigma_1 \otimes \sigma_0, \sigma_0 \otimes \sigma_3, \sigma_1 \otimes \sigma_3\}\}$	$\emptyset$
$\widetilde{\mathcal{G}}_8^{\mathbb{R}} \subsetneqq \mathcal{G}_7^{\mathbb{R}}$	$\{A \in \{\sigma_0 \otimes \sigma_0, \sigma_3 \otimes \sigma_0, \sigma_0 \otimes \sigma_1, \sigma_3 \otimes \sigma_1\}\}$	$C = I_4$

TABLE 13. List of all non-conjugate real parts $\mathcal{G}_j^{\mathbb{R}}$ of MAD-groups $\mathcal{G}_j$ on $sl(4,\mathbb{C})$. The first six items are the maximal ones ($\mathcal{G}_1^{\mathbb{R}}$, $\mathcal{G}_3^{\mathbb{R}}$, $\mathcal{G}_4^{\mathbb{R}}$, $\mathcal{G}_5^{\mathbb{R}}$, $\mathcal{G}_6^{\mathbb{R}}$, $\mathcal{G}_7^{\mathbb{R}}$), while the last two are not maximal ($\mathcal{G}_2^{\mathbb{R}}$, and $\widetilde{\mathcal{G}}_8^{\mathbb{R}}$ - already in the conjugate form, so as to see the link with $\mathcal{G}_7^{\mathbb{R}}$ easily). The notation of G_{Ad} and G_{Out} is the same as introduced in Table 1.

From [8], we have a result summarizing which of the real forms $\mathcal{G}_j^{\mathbb{R}}$ are in fact MAD-groups on the individual real forms of $sl(4,\mathbb{C})$ - see the overview in Table 14.

real form	MAD-groups of the real form
$sl(4,\mathbb{R})$	$\mathcal{G}_1^{\mathbb{R}}, \mathcal{G}_3^{\mathbb{R}}, \mathcal{G}_4^{\mathbb{R}}, \mathcal{G}_5^{\mathbb{R}}, \mathcal{G}_6^{\mathbb{R}}, \mathcal{G}_7^{\mathbb{R}}$
$su^*(4)$	$\mathcal{G}_6^{\mathbb{R}}, \mathcal{G}_7^{\mathbb{R}}$
$su(4,0)$	$\mathcal{G}_3^{\mathbb{R}}, \mathcal{G}_7^{\mathbb{R}}$
$su(3,1)$	$\mathcal{G}_3^{\mathbb{R}}, \mathcal{G}_4^{\mathbb{R}}$
$su(2,2)$	$\mathcal{G}_3^{\mathbb{R}}, \mathcal{G}_4^{\mathbb{R}}, \mathcal{G}_5^{\mathbb{R}}, \mathcal{G}_6^{\mathbb{R}}, \mathcal{G}_7^{\mathbb{R}}$

TABLE 14. MAD-groups on the five real forms of $sl(4,\mathbb{C})$

Now that we dispose of the full list of MAD-groups on all the five real forms, we can apply the 'MAD-group' method - i.e. decompose the real forms into simultaneous eigensubspaces of all

elements of the respective MAD-groups, and by that obtain the fine group gradings. We do not have here the one-to-one correspondence as guaranteed for the complex Lie algebras by Theorem 1, however, we can at least make use of a weaker statement (proved in [8]):

Theorem 4. *'MAD-group' method: Let $L_{\mathcal{J}}$ be a real form of a classical complex Lie algebra L, $L \neq o(8,\mathbb{C})$. Let $\mathcal{F}$ be a MAD-group on $L_{\mathcal{J}}$. Then the grading of $L_{\mathcal{J}}$ generated by $\mathcal{F}$ is a fine group grading.*

This theorem ensures that each MAD-group on the real form generates a fine group grading, however, it does not ensure the opposite implication - namely that this method provides all the fine group gradings of the real forms. And indeed, we manage to prove that there exist more fine group gradings of the real forms of $sl(4,\mathbb{C})$ than those generated by MAD-groups, simply by finding counterexamples of additional fine group gradings that are not generated by any MAD-group of the real form.

We do this by using the following method - let us call it the 'Fundamental' method. We work with the basis vectors of the grading subspaces of the complex gradings and transform them, so as to obtain a fine group grading of the real form; more concretely:

Theorem 5. *'Fundamental' method: Let $\Gamma : L = \oplus_{k\in\mathcal{K}} L_k$ be a fine group grading of the complex Lie algebra L. Let $\mathcal{J}$ be an involutive antiautomorphism on L, let $L_{\mathcal{J}}$ be the real form of L defined by $\mathcal{J}$, and let $Z_{k,l}$ be elements of L_k fulfilling the following properties:*

- *$(Z_{k,1},\dots,Z_{k,l_k})$ is a basis of the grading subspace L_k for each $k \in \mathcal{K}$ - i.e. $L_k = \mathrm{span}^{\mathbb{C}}(Z_{k,1},\dots,Z_{k,l_k})$;*
- *$\mathcal{J}(Z_{k,l}) = Z_{k,l}$ - i.e. $Z_{k,l} \in L_{\mathcal{J}}$ for all $Z_{k,l} \in L_k$.*

Then the decomposition $\Gamma^{\mathcal{J}} : L_{\mathcal{J}} = \oplus_{k\in\mathcal{K}} L_k^{\mathbb{R}}$, where $L_k^{\mathbb{R}} = \mathrm{span}^{\mathbb{R}}(Z_{k,1},\dots,Z_{k,l_k})$, is a fine group grading of the real form $L_{\mathcal{J}}$.
We then say that the grading Γ of the complex Lie algebra L **determines** *the fine group grading $\Gamma^{\mathcal{J}}$ of the real form $L_{\mathcal{J}}$.*

Note that the same statement is valid also when considering fine gradings instead of fine group gradings.

The assumptions imposed on the grading subspaces L_k are formulated as two conditions for the basis vectors of these grading subspaces. This algorithmic approach can be modified into just one requirement in the form $L_{\mathcal{J}} = \oplus_{k\in\mathcal{K}} L_k \cap L_{\mathcal{J}}$. In the notation of Theorem 5, the subspace $L_k^{\mathbb{R}}$ equals to $L_k \cap L_{\mathcal{J}}$. This expression clearly shows that the fine group grading of $L_{\mathcal{J}}$ obtained in this way from the fine group grading of L is unique.

Practically, the usage of this method consists in the following process:

- For a given fine group grading Γ of $sl(4,\mathbb{C})$, we search for an involutive antiautomorphism h in the form $h = \mathcal{J}_0 Out_E$ or $h = \mathcal{J}_0 Ad_F$, with matrices E, F fulfilling the requirements listed in Table 12, such that there exists a basis $(Z_{k,1},\dots,Z_{k,l_k})$ of L_k with $\mathcal{J}(Z_{k,l}) = Z_{k,l}$ - as required in Theorem 5. If successful, this process results in finding the fine group gradings of as many of the real forms of $sl(4,\mathbb{C})$ as possible.
- We repeat this process for all the fine group gradings Γ_j of $sl(4,\mathbb{C})$.

We need to keep in mind that various fine group gradings Γ_j of $sl(4,\mathbb{C})$ determine fine group gradings of one real form in different representations, depending on the defining matrix E or F found during the process.

As an illustration of the calculations carried out by the 'Fundamental' method, we give an example of how we found out which fine group gradings of real forms of $sl(4,\mathbb{C})$ are determined by the fine grading Γ_2 of $sl(4,\mathbb{C})$. Even though the real part $\mathcal{G}_2^{\mathbb{R}}$ of the MAD-group $\mathcal{G}_2$, which generates Γ_2, does not provide MAD-group on any of the real forms of $sl(4,\mathbb{C})$, we show that this grading Γ_2 does determine a fine group grading on $su(3,1)$ and on $su(2,2)$, and does not determine fine group gradings on the remaining real forms of $sl(4,\mathbb{C})$.

Firstly, let us inspect the cases $su(4-r,r)$. Since the grading subspaces L_k of the grading Γ_2 are one-dimensional, and thus $L_k = \mathbb{C}\cdot X_k$, we need to find the convenient basis of L_k for the Theorem 5 in the form $Z_k = \alpha_k X_k$, with $\alpha_k \in \mathbb{C}\setminus\{0\}$. We look for a hermitian matrix E ($E = E^{+}$) with $4-r$ positive and r negative eigenvalues such that, for all X_k from the basis listed in Table 3, there exists $\alpha_k \in \mathbb{C}\setminus\{0\}$ fulfilling the equation

$$\mathcal{J}(\alpha_k X_k) = \mathcal{J}_0 Out_E(\alpha_k X_k) = \alpha_k X_k \qquad \Leftrightarrow \qquad \alpha_k X_k E = -\overline{\alpha_k} E X_k^{+}. \tag{4}$$

Let us start with $X_4 = P = \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0 \end{pmatrix}$. Denoting $\eta = \frac{\overline{\alpha_4}}{\alpha_4}$, we obtain from (4) that the matrix E has the form $E = \begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14}\\ -\eta e_{12} & -\eta e_{13} & -\eta e_{14} & -\eta e_{11}\\ \eta^2 e_{13} & \eta^2 e_{14} & \eta^2 e_{11} & \eta^2 e_{12}\\ -\eta^3 e_{14} & -\eta^3 e_{11} & -\eta^3 e_{12} & -\eta^3 e_{13} \end{pmatrix}$, and $\eta^4 = 1$. We continue with $X_3 = Q = \mathrm{diag}(1, i, -1, -i)$ and we find out that the following four equations must hold so that the condition (4) could be fulfilled:

$$\begin{aligned} (\alpha_3 + \overline{\alpha_3})e_{11} &= 0\\ (\alpha_3 - i\overline{\alpha_3})e_{12} &= 0\\ (\alpha_3 - \overline{\alpha_3})e_{13} &= 0\\ (\alpha_3 + i\overline{\alpha_3})e_{14} &= 0 \end{aligned} \tag{5}$$

This set of equations implies that at most one of the four elements $e_{11}, e_{12}, e_{13}, e_{14}$ is non-zero. On the other hand, at least one of them must be non-zero, otherwise the matrix E would be a zero matrix, and thus it would not have $4-r$ positive and r negative eigenvalues. So we have four possibilities and we will go through them gradually:

- Let $e_{11} \neq 0$. It follows from (5) that $e_{12} = e_{13} = e_{14} = 0$. Then the matrix E equals $E = e_{11}\begin{pmatrix} 1&0&0&0\\ 0&0&0&-\eta\\ 0&0&\eta^2&0\\ 0&-\eta^3&0&0 \end{pmatrix}$. Spectrum of E is $\sigma(E) = \{e_{11}, e_{11}, -e_{11}, \eta^2 e_{11}\}$, $e_{11} \in \mathbb{R}\setminus\{0\}$, since E must be hermitian. So there is at least one positive and one negative eigenvalue in $\sigma(E)$. Thus $\mathcal{J}_0 Out_E$ with this type of E does not define the real form $su(4,0)$. But for $su(3,1)$ and $su(2,2)$ we are successful:
  - With $e_{11} = 1$, $\alpha_4 = i$, $\eta^2 = 1$ the spectrum of $E_3^{(3,1)} = \begin{pmatrix} 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0 \end{pmatrix}$ is equal to $\sigma(E_3^{(3,1)}) = \{1,1,1,-1\}$. Therefore, the fine group grading Γ_2 of $sl(4,\mathbb{C})$ determines a fine group grading of $su(3,1) = L_{\mathcal{J}_0 Out_{E_3^{(3,1)}}}$.
  - Putting $e_{11} = 1$, $\alpha_4 = 1-i$, $\eta^2 = -1$, the spectrum of $E_4^{(2,2)} = \begin{pmatrix} 1&0&0&0\\ 0&0&0&-i\\ 0&0&-1&0\\ 0&i&0&0 \end{pmatrix}$ is equal to $\sigma(E_4^{(2,2)}) = \{1,1,-1,-1\}$. Therefore, the fine group grading Γ_2 of $sl(4,\mathbb{C})$ determines a fine group grading of $su(2,2) = L_{\mathcal{J}_0 Out_{E_4^{(2,2)}}}$.
- Let $e_{12} \neq 0$. Then $e_{11} = e_{13} = e_{14} = 0$,

$$E = e_{12}\begin{pmatrix} 0&1&0&0\\ -\eta&0&0&0\\ 0&0&0&\eta^2\\ 0&0&-\eta^3&0 \end{pmatrix}, \tag{6}$$

  and $\sigma(E) = \{\pm i\sqrt{\eta}e_{12}, \pm i\sqrt{\eta}e_{12}\}$. Every hermitian matrix of this type has two positive and two negative eigenvalues, but $su(2,2)$ is already covered by the previous type. The important fact is that we cannot obtain any hermitian matrix E of the type (6) with positive spectrum.
- Let $e_{13} \neq 0$. Then $e_{11} = e_{12} = e_{14} = 0$, $E = e_{13}\begin{pmatrix} 0&0&1&0\\ 0&-\eta&0&0\\ \eta^2&0&0&0\\ 0&0&0&-\eta^3 \end{pmatrix}$, and $\sigma(E) = \{\pm\eta e_{13}, -\eta e_{13}, -\eta^3 e_{13}\}$. Again this kind of matrix E does not admit positive spectrum.
- Let $e_{14} \neq 0$. Then $e_{11} = e_{12} = e_{13} = 0$, $E = e_{14}\begin{pmatrix} 0&0&0&1\\ 0&0&-\eta&0\\ 0&\eta^2&0&0\\ -\eta^3&0&0&0 \end{pmatrix}$, $\sigma(E) = \{\pm i\eta\sqrt{\eta}e_{14}, \pm i\eta\sqrt{\eta}e_{14}\}$, which, again, does not admit positive spectrum for any matrix E.

One can see that none of the four possibilities allows a hermitian matrix E with positive spectrum. Therefore Γ_2 does not determine fine group grading on any representation of the real form $su(4,0)$.

Now we move to the two real forms $sl(4,\mathbb{R})$ and $su^*(4)$. The real form $sl(4,\mathbb{R})$ (resp. $su^*(4)$) has a fine group grading determined by Γ_2 if there exists a circular (resp. anticircular) matrix F such that, for each X_k from the basis listed in Table 3, there exists $\alpha_k \in \mathbb{C}\setminus\{0\}$ fulfilling

$$\mathcal{J}(\alpha_k X_k) = \mathcal{J}_0 Ad_F(\alpha_k X_k) = \alpha_k X_k \qquad \Leftrightarrow \qquad \alpha_k X_k F = \overline{\alpha_k} F \overline{X_k}. \tag{7}$$

For $X_4 = P$ the condition (7) implies that F has the form

$$F = \begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ \nu f_{14} & \nu f_{11} & \nu f_{12} & \nu f_{13} \\ \nu^2 f_{13} & \nu^2 f_{14} & \nu^2 f_{11} & \nu^2 f_{12} \\ \nu^3 f_{12} & \nu^3 f_{13} & \nu^3 f_{14} & \nu^3 f_{11} \end{pmatrix}, \tag{8}$$

where $\nu = \frac{\overline{\alpha_4}}{\alpha_4}$. Again we proceed with inserting $X_3 = Q$ into the equation (7). Denoting $\mu = \frac{\overline{\alpha_3}}{\alpha_3}$, we arrive at the following set of equations:

$$\begin{array}{rclcrcl} (1-\mu)f_{11} & = & 0 & \qquad & (1+\mu)f_{11} & = & 0 \\ (1-i\mu)f_{12} & = & 0 & & (1+i\mu)f_{12} & = & 0 \\ (1-\mu)f_{13} & = & 0 & & (1+\mu)f_{13} & = & 0 \\ (1-i\mu)f_{14} & = & 0 & & (1+i\mu)f_{14} & = & 0. \end{array} \tag{9}$$

Clearly (9) can only be fulfilled when $f_{11} = f_{12} = f_{13} = f_{14} = 0$. But it means that F would be a zero matrix. And a zero matrix Θ is not circular: $F\overline{F} = \Theta\overline{\Theta} = \Theta \neq I$. Thus from Γ_2 one does not obtain any fine group grading of $sl(4,\mathbb{R})$. The situation with $su^*(4)$ is analogous - by the same process we come to the conclusion that $F = \Theta$, and a zero matrix cannot be anticircular ($F\overline{F} = \Theta\overline{\Theta} = \Theta \neq -I$). Neither $su^*(4)$ has any fine group grading determined by Γ_2.

Apart from Theorem 5, we can use one more powerful tool, which simplifies the process of the 'Fundamental' method for those fine gradings Γ of $sl(4,\mathbb{C})$ whose grading subspaces have basis matrices with real elements only, and that is the case for Γ_j, $j = 1,3,4,5,6,7,8$.

Theorem 6. *'Real Basis' method: Let $\mathcal{G}$ be a MAD-group on the complex Lie algebra $sl(n,\mathbb{C})$ and let $\Gamma : sl(n,\mathbb{C}) = \oplus_{k\in\mathcal{K}} L_k$ be the fine group grading of $sl(n,\mathbb{C})$ generated by $\mathcal{G}$ such that all the subspaces $L_k = \mathrm{span}^{\mathbb{C}}(X_{k,1},\ldots,X_{k,l_k})$ have real basis vectors $X_{k,l}$. Let h be an automorphism in $\mathcal{A}ut\, sl(n,\mathbb{C})$, such that $\mathcal{J} = \mathcal{J}_0 h$ is an involutive antiautomorphism on $sl(n,\mathbb{C})$. Then the fine group grading Γ of $sl(n,\mathbb{C})$ determines a fine group grading of the real form $L_{\mathcal{J}} = L_{\mathcal{J}_0 h}$ if and only if the automorphism h is an element of the MAD-group $\mathcal{G}$.*

Proof. Firstly, let us consider an automorphism $h \in \mathcal{G}$. The automorphism h is either of the type $h = Ad_F$ with F circular or anticircular ($F\overline{F} = \pm I$), or $h = Out_E$ with E hermitian ($E = E^+$). Since $\mathcal{G}$ is the MAD-group generating the group grading Γ, any $X_{k,l} \in L_k$ is an eigenvector of h. We show that, for both the types of h and for any of the basis vectors $X_{k,l}$, the eigenvalue λ_k of the eigenvector $X_{k,l}$ has its absolute value equal to 1:

- inner automorphism $h = Ad_F$:
  $\lambda X = h(X) = F^{-1}XF = \overline{F}X\overline{F}^{-1} = \overline{FXF^{-1}} = \overline{F(\frac{1}{\lambda}F^{-1}XF)F^{-1}} = \frac{1}{\overline{\lambda}}X$
- outer automorphism $h = Out_E$:
  $\lambda X = h(X) = -(E^{-1}XE)^T = -E^T X^T E^{-T} = -\overline{E^+ X^T (E^+)^{-1}} =$
  $= -\overline{EX^T E^{-1}} = -\overline{E(-\frac{1}{\lambda}E^{-1}XE)E^{-1}} = \frac{1}{\overline{\lambda}}X$

The basis vectors $X_{k,l}$ are non-zero, therefore $\lambda_k X_{k,l} = \frac{1}{\overline{\lambda_k}} X_{k,l}$ implies that $|\lambda_k| = 1$. Thus we can express λ_k as $\lambda_k = \cos\varphi_k + i\sin\varphi_k$. We define a new basis vector $Z_{k,l} = \alpha_k X_{k,l}$, where $\alpha_k = \cos(-\frac{\varphi_k}{2}) + i\sin(-\frac{\varphi_k}{2})$. (Such choice of α_k ensures the property $\lambda_k = \frac{\overline{\alpha_k}}{\alpha_k}$.) This basis vector $Z_{k,l}$ is an element of $L_{\mathcal{J}_0 h}$: $\mathcal{J}_0 h(Z_{k,l}) = \mathcal{J}_0 h(\alpha_k X_{k,l}) = \mathcal{J}_0(\alpha_k h(X_{k,l})) = \mathcal{J}_0(\alpha_k \lambda_k X_{k,l}) = \mathcal{J}_0(\alpha_k \frac{\overline{\alpha_k}}{\alpha_k} X_{k,l}) = \mathcal{J}_0(\overline{\alpha_k} X_{k,l}) = \alpha_k \mathcal{J}_0(X_{k,l}) = \alpha_k \overline{X_{k,l}} = \alpha_k X_{k,l} = Z_{k,l}$. By Theorem 5 ('Fundamental' method) it follows that the sum $L_{\mathcal{J}_0 h} = \oplus_{k\in\mathcal{K}} L_k^{\mathbb{R}}$ with subspaces $L_k^{\mathbb{R}} = \mathrm{span}^{\mathbb{R}}(Z_{k,1},\ldots,Z_{k,l_k})$ is a fine group grading of $L_{\mathcal{J}_0 h}$.

Secondly, we assume that Γ determines a fine group grading of the real form $L_{\mathcal{J}} = L_{\mathcal{J}_0 h}$ with an automorphism $h \in \mathcal{A}ut\, sl(n,\mathbb{C})$. We start by showing that h commutes with any element of the MAD-group $\mathcal{G}$. Notice that $\mathcal{J}_0(L_k) = L_k$ by hypothesis; and let $g \in \mathcal{G}$. It follows from the definition of the grading Γ that for any $k \in \mathcal{K}$ there exists $\alpha_k \in \mathbb{C}$ such that $g\mid_{L_k} = \alpha_k Id$. In order to prove that $gh = gh$, it is sufficient to show that $hg\mid_{L_k} = gh\mid_{L_k}$ for any $k \in \mathcal{K}$. Clearly $L_{\mathcal{J}} = \oplus(L_k \cap L_{\mathcal{J}})$, and therefore $\mathbb{C}\cdot(L_k \cap L_{\mathcal{J}}) = L_k$. Thus, for any $x \in L_k$ there are $y, z \in L_k \cap L_{\mathcal{J}}$ such that $x = y + iz$. In particular, $\mathcal{J}(y) = y$ and $h(y) = \overline{y}$, and analogously for z. It follows that $h(x) = h(y) + ih(z) = \overline{y} + i\overline{z}$. Knowing that $\overline{y}, \overline{z} \in \mathcal{J}_0(L_k) = L_k$, we apply g and obtain $gh(x) = \alpha_k(\overline{y - iz})$. Finally, the action of g on x by $g(x) = \alpha_k x$ ensures that $hg(x) = h(\alpha_k x) = \alpha_k h(x) = \alpha_k(\overline{y - iz})$, which means that g and h commute. Then, when checking in [7] the form of the MAD-groups on $sl(n,\mathbb{C})$, one derives that any automorphism which commutes

with the whole MAD-group must necessarily be diagonalizable. And that already implies, due to the maximality of $\mathcal{G}$, that $h \in \mathcal{G}$. □

Let us now summarize the results of our calculations on the real forms of $sl(4,\mathbb{C})$; partially obtained with the use of Theorem 5 - 'Fundamental' method (for Γ_2), partially with the additional help of Theorem 6 - 'Real Basis' method (for Γ_8). In Table 15, we list the various representations of the real forms that will be used in the result summary.

Defining matrix $F^{(4,\mathbb{R})}$	Representation of $sl(4,\mathbb{R}) = L_{\mathcal{J}_0 Ad_{F^{(4,\mathbb{R})}}}$
$F_1^{(4,\mathbb{R})} = I_4$	$X = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ j & k & l & m \\ n & o & p & -(a+f+l) \end{pmatrix}$
Defining matrix $F^{*(4)}$	**Representation of $su^*(4) = L_{\mathcal{J}_0 Ad_{F^{*(4)}}}$**
$F_1^{*(4)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$	$X = \begin{pmatrix} a+il & b+im & c+ij & d+ik \\ e+ip & -a+if & g+in & h+io \\ -c+ij & -d+ik & a-il & b-im \\ -g+in & -h+io & l-ip & -a-if \end{pmatrix}$
$F_2^{*(4)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}$	$X = \begin{pmatrix} a+if & b+ie & c+ih & d+ig \\ -b+ie & a-if & id+g & -ic-h \\ j+io & k+in & -a+il & m+ip \\ -ik-n & ij+o & -m+ip & -a-il \end{pmatrix}$
Defining matrix $E^{(4,0)}$	**Representation of $su(4,0) = L_{\mathcal{J}_0 Out_{E^{(4,0)}}}$**
$E_1^{(4,0)} = I_4$	$X = \begin{pmatrix} ia & b+ie & c+ij & d+in \\ -b+ie & if & g+ik & h+io \\ -c+ij & -g+ik & il & m+ip \\ -d+in & -h+io & -m+ip & -i(a+f+l) \end{pmatrix}$
Defining matrix $E^{(3,1)}$	**Representation of $su(3,1) = L_{\mathcal{J}_0 Out_{E^{(3,1)}}}$**
$E_1^{(3,1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$X = \begin{pmatrix} ia & b+ie & c+ij & d+in \\ -b+ie & if & g+ik & h+io \\ -c+ij & -g+ik & il & m+ip \\ d-in & h-io & m-ip & -i(a+f+l) \end{pmatrix}$
$E_2^{(3,1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$X = \begin{pmatrix} ia & b+ie & c+in & d+ij \\ -b+ie & if & g+io & h+ik \\ -d+ij & -h+ik & l-\frac{i}{2}(a+f) & im \\ -c+in & -g+io & ip & -l-\frac{i}{2}(a+f) \end{pmatrix}$
$E_3^{(3,1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$X = \begin{pmatrix} ia & b+in & c+ij & d+ie \\ -d+ie & f-\frac{i}{2}(a+l) & g+im & ih \\ -c+ij & k+ip & il & -g+im \\ -b+in & io & -k+ip & -f-\frac{i}{2}(a+l) \end{pmatrix}$
Defining matrix $E^{(2,2)}$	**Representation of $su(2,2) = L_{\mathcal{J}_0 Out_{E^{(2,2)}}}$**
$E_1^{(2,2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$X = \begin{pmatrix} ia & b+ie & c+ij & d+in \\ -b+ie & if & g+ik & h+io \\ c-ij & g-ik & il & m+ip \\ d-in & h-io & -m+ip & -i(a+f+l) \end{pmatrix}$
$E_2^{(2,2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$X = \begin{pmatrix} ia & b+ie & c+in & d+ij \\ b-ie & if & g+io & h+ik \\ -d+ij & -h+ik & l-\frac{i}{2}(a+f) & im \\ -c+in & g-io & ip & -l-\frac{i}{2}(a+f) \end{pmatrix}$
$E_3^{(2,2)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$X = \begin{pmatrix} a+if & ib & c+io & d+ik \\ ie & -a+if & g+in & h+ij \\ -h+ij & -d+ik & l-if & im \\ -g+in & -c+io & ip & -l-if \end{pmatrix}$
$E_4^{(2,2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -1 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$	$X = \begin{pmatrix} ia & b+in & c+ij & d+ie \\ id+e & f-\frac{i}{2}(a+l) & g+im & h \\ c-ij & k+ip & il & -ig-m \\ -ib-n & o & ik+p & -f-\frac{i}{2}(a+l) \end{pmatrix}$
$E_5^{(2,2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}$	$X = \begin{pmatrix} ia & b+ie & c+in & d+ij \\ b-ie & if & g+io & h+ik \\ -id-j & ih+k & l-\frac{i}{2}(a+f) & m \\ ic+n & -ig-o & p & -l-\frac{i}{2}(a+f) \end{pmatrix}$

TABLE 15. This is the summary of all the various representations of the five real forms of $sl(4,\mathbb{C})$ that we need in order to describe the fine group gradings of these five real forms. For each of the representations, a generic element X is expressed, assuming the parameters $a, \ldots, p$ to be real.

The full overview of the fine group gradings of the five real forms of $sl(4,\mathbb{C})$ determined by the individual fine gradings Γ_j of $sl(4,\mathbb{C})$ is contained in Tables 16, …, 23. We proceed gradually with $\Gamma_1, \Gamma_2, \ldots, \Gamma_8$, and we describe the basis vectors of the grading subspaces in suitable representation

of the graded real form. We use the same sets of matrices $X_1, \ldots, X_{15}$ as already listed in Section 3.1.

Γ_1 grading subspaces	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8
	L_9	L_{10}	L_{11}	L_{12}	L_{13}			
real form $sl(4,\mathbb{R})$	$\mathbb{R}X_1$	$\mathbb{R}X_2$	$\mathbb{R}X_3$	$\mathbb{R}X_4$	$\mathbb{R}X_5$	$\mathbb{R}X_6$	$\mathbb{R}X_7$	$\mathbb{R}X_8$
defined by $F_1^{(4,\mathbb{R})}$	$\mathbb{R}X_9$	$\mathbb{R}X_{10}$	$\mathbb{R}X_{11}$	$\mathbb{R}X_{12}$	$\mathbb{R}X_{13}+\mathbb{R}X_{14}+\mathbb{R}X_{15}$			

TABLE 16. The fine group grading Γ_1 of $sl(4,\mathbb{C})$ determines a fine group grading $\Gamma_1^{\mathcal{J}}$ for just one real form of $sl(4,\mathbb{C})$, namely for $L_{\mathcal{J}} = sl(4,\mathbb{R})$. The basis vectors X_k are those listed in Table 2.

Γ_2 grading subspaces	L_1	L_2	L_3	L_4	L_5
	L_6	L_7	L_8	L_9	L_{10}
	L_{11}	L_{12}	L_{13}	L_{14}	L_{15}
real form $su(3,1)$	$i\mathbb{R}X_1$	$i\mathbb{R}X_2$	$i\mathbb{R}X_3$	$i\mathbb{R}X_4$	$(1-i)\mathbb{R}X_5$
defined by $E_3^{(3,1)}$	$\mathbb{R}X_6$	$(1+i)\mathbb{R}X_7$	$i\mathbb{R}X_8$	$\mathbb{R}X_9$	$i\mathbb{R}X_{10}$
	$\mathbb{R}X_{11}$	$i\mathbb{R}X_{12}$	$(1+i)\mathbb{R}X_{13}$	$\mathbb{R}X_{14}$	$(1-i)\mathbb{R}X_{15}$
real form $su(2,2)$	$i\mathbb{R}X_1$	$i\mathbb{R}X_2$	$i\mathbb{R}X_3$	$(1-i)\mathbb{R}X_4$	$\mathbb{R}X_5$
defined by $E_4^{(2,2)}$	$(1+i)\mathbb{R}X_6$	$i\mathbb{R}X_7$	$\mathbb{R}X_8$	$i\mathbb{R}X_9$	$\mathbb{R}X_{10}$
	$i\mathbb{R}X_{11}$	$(1+i)\mathbb{R}X_{12}$	$\mathbb{R}X_{13}$	$(1-i)\mathbb{R}X_{14}$	$i\mathbb{R}X_{15}$

TABLE 17. The fine group grading Γ_2 of $sl(4,\mathbb{C})$ determines a fine group grading $\Gamma_2^{\mathcal{J}}$ for two real forms of $sl(4,\mathbb{C})$, namely for $L_{\mathcal{J}} = su(3,1)$ and $L_{\mathcal{J}} = su(2,2)$. This result cannot be obtained via Theorem 6, as the bases X_k of the grading subspaces in Γ_2 are not real matrices - see Table 3.

Γ_3 grading subspaces	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8
	L_9	L_{10}	L_{11}	L_{12}	L_{13}			
real form $sl(4,\mathbb{R})$	$\mathbb{R}X_1$	$\mathbb{R}X_2$	$\mathbb{R}X_3$	$\mathbb{R}X_4$	$\mathbb{R}X_5$	$\mathbb{R}X_6$	$\mathbb{R}X_7$	$\mathbb{R}X_8$
defined by $F_1^{(4,\mathbb{R})}$	$\mathbb{R}X_9$	$\mathbb{R}X_{10}$	$\mathbb{R}X_{11}$	$\mathbb{R}X_{12}$	$\mathbb{R}X_{13}+\mathbb{R}X_{14}+\mathbb{R}X_{15}$			
real form $su(4,0)$	$i\mathbb{R}X_1$	$\mathbb{R}X_2$	$i\mathbb{R}X_3$	$\mathbb{R}X_4$	$i\mathbb{R}X_5$	$\mathbb{R}X_6$	$i\mathbb{R}X_7$	$\mathbb{R}X_8$
defined by $E_1^{(4,0)}$	$i\mathbb{R}X_9$	$\mathbb{R}X_{10}$	$i\mathbb{R}X_{11}$	$\mathbb{R}X_{12}$	$i\mathbb{R}X_{13}+i\mathbb{R}X_{14}+i\mathbb{R}X_{15}$			
real form $su(3,1)$	$\mathbb{R}X_1$	$i\mathbb{R}X_2$	$\mathbb{R}X_3$	$i\mathbb{R}X_4$	$\mathbb{R}X_5$	$i\mathbb{R}X_6$	$i\mathbb{R}X_7$	$\mathbb{R}X_8$
defined by $E_1^{(3,1)}$	$i\mathbb{R}X_9$	$\mathbb{R}X_{10}$	$i\mathbb{R}X_{11}$	$\mathbb{R}X_{12}$	$i\mathbb{R}X_{13}+i\mathbb{R}X_{14}+i\mathbb{R}X_{15}$			
real form $su(2,2)$	$\mathbb{R}X_1$	$i\mathbb{R}X_2$	$\mathbb{R}X_3$	$i\mathbb{R}X_4$	$i\mathbb{R}X_5$	$\mathbb{R}X_6$	$\mathbb{R}X_7$	$i\mathbb{R}X_8$
defined by $E_1^{(2,2)}$	$\mathbb{R}X_9$	$i\mathbb{R}X_{10}$	$i\mathbb{R}X_{11}$	$\mathbb{R}X_{12}$	$i\mathbb{R}X_{13}+i\mathbb{R}X_{14}+i\mathbb{R}X_{15}$			

TABLE 18. Four of the real forms of $sl(4,\mathbb{C})$ - i.e. $L_{\mathcal{J}} = sl(4,\mathbb{R})$, $L_{\mathcal{J}} = su(4,0)$, $L_{\mathcal{J}} = su(3,1)$, and $L_{\mathcal{J}} = su(2,2)$ - have a fine group grading $\Gamma_3^{\mathcal{J}}$ determined by the fine group grading Γ_3 of $sl(4,\mathbb{C})$; with basis vectors being multiples of X_k defined in Table 4 by complex coefficients as listed here.

Comparing these results with the ones obtained by the 'MAD-group' method (exhausting the whole list of MAD-groups $\mathcal{G}_1^{\mathbb{R}}$, $\mathcal{G}_3^{\mathbb{R}}$, $\mathcal{G}_4^{\mathbb{R}}$, $\mathcal{G}_5^{\mathbb{R}}$, $\mathcal{G}_6^{\mathbb{R}}$, and $\mathcal{G}_7^{\mathbb{R}}$ acting on one or more of the real forms), we see that the 'real basis' method and the more general 'fundamental' method bring in six additional fine group gradings of the real forms, namely those determined by the gradings Γ_2 and Γ_8 of $sl(4,\mathbb{C})$. The fact that for the complex MAD-groups $\mathcal{G}_2$ and $\mathcal{G}_8$ there exist no MAD-groups on real forms can be deduced from the universal grading groups $\mathbb{Z}_4^2$ and $\mathbb{Z}_4 \times \mathbb{Z}_2^2$ corresponding to the

Γ_4 grading	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8
subspaces	L_9	L_{10}	L_{11}	L_{12}	L_{13}	L_{14}		
real form $sl(4,\mathbb{R})$	$\mathbb{R}X_1$	$\mathbb{R}X_2$	$\mathbb{R}X_3$	$\mathbb{R}X_4$	$\mathbb{R}X_5$	$\mathbb{R}X_6$	$\mathbb{R}X_7$	$\mathbb{R}X_8$
defined by $F_1^{(4,\mathbb{R})}$	$\mathbb{R}X_9$	$\mathbb{R}X_{10}$	$\mathbb{R}X_{11}$	$\mathbb{R}X_{12}$	$\mathbb{R}X_{13}$	$\mathbb{R}X_{14}+\mathbb{R}X_{15}$		
real form $su(3,1)$	$\mathbb{R}X_1$	$i\mathbb{R}X_2$	$\mathbb{R}X_3$	$i\mathbb{R}X_4$	$\mathbb{R}X_5$	$i\mathbb{R}X_6$	$\mathbb{R}X_7$	$i\mathbb{R}X_8$
defined by $E_2^{(3,1)}$	$\mathbb{R}X_9$	$i\mathbb{R}X_{10}$	$i\mathbb{R}X_{11}$	$i\mathbb{R}X_{12}$	$\mathbb{R}X_{13}$	$i\mathbb{R}X_{14}+i\mathbb{R}X_{15}$		
real form $su(2,2)$	$\mathbb{R}X_1$	$i\mathbb{R}X_2$	$\mathbb{R}X_3$	$i\mathbb{R}X_4$	$i\mathbb{R}X_5$	$\mathbb{R}X_6$	$i\mathbb{R}X_7$	$\mathbb{R}X_8$
defined by $E_2^{(2,2)}$	$i\mathbb{R}X_9$	$\mathbb{R}X_{10}$	$i\mathbb{R}X_{11}$	$i\mathbb{R}X_{12}$	$\mathbb{R}X_{13}$	$i\mathbb{R}X_{14}+i\mathbb{R}X_{15}$		

TABLE 19. The fine group grading Γ_4 of $sl(4,\mathbb{C})$ determines a fine group grading $\Gamma_4^{\mathcal{J}}$ for three of the real forms of $sl(4,\mathbb{C})$: $L_{\mathcal{J}} = sl(4,\mathbb{R})$, $L_{\mathcal{J}} = su(3,1)$, and $L_{\mathcal{J}} = su(2,2)$. Their bases can be obtained from matrices X_k defined in Table 5 via multiplication by complex coefficients indicated here.

Γ_5 grading	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8
subspaces	L_9	L_{10}	L_{11}	L_{12}	L_{13}	L_{14}		
real form $sl(4,\mathbb{R})$	$\mathbb{R}X_1$	$\mathbb{R}X_2$	$\mathbb{R}X_3$	$\mathbb{R}X_4$	$\mathbb{R}X_5$	$\mathbb{R}X_6$	$\mathbb{R}X_7$	$\mathbb{R}X_8$
defined by $F_1^{(4,\mathbb{R})}$	$\mathbb{R}X_9$	$\mathbb{R}X_{10}$	$\mathbb{R}X_{11}$	$\mathbb{R}X_{12}$	$\mathbb{R}X_{13}$	$\mathbb{R}X_{14}+\mathbb{R}X_{15}$		
real form $su(2,2)$	$\mathbb{R}X_1$	$i\mathbb{R}X_2$	$\mathbb{R}X_3$	$i\mathbb{R}X_4$	$\mathbb{R}X_5$	$i\mathbb{R}X_6$	$\mathbb{R}X_7$	$i\mathbb{R}X_8$
defined by $E_3^{(2,2)}$	$i\mathbb{R}X_9$	$i\mathbb{R}X_{10}$	$i\mathbb{R}X_{11}$	$i\mathbb{R}X_{12}$	$i\mathbb{R}X_{13}$	$\mathbb{R}X_{14}+\mathbb{R}X_{15}$		

TABLE 20. The fine group grading Γ_5 of $sl(4,\mathbb{C})$ determines a fine group grading $\Gamma_5^{\mathcal{J}}$ for the real forms $L_{\mathcal{J}} = sl(4,\mathbb{R})$ and $L_{\mathcal{J}} = su(2,2)$ of the complex algebra $sl(4,\mathbb{C})$. The matrices X_k listed in Table 6 are to be multiplied by complex coefficients from here, in order to obtain the basis matrices for the fine group grading of the respective real form.

Γ_6 grading	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8
subspaces	L_9	L_{10}	L_{11}	L_{12}	L_{13}	L_{14}	L_{15}	
real form $sl(4,\mathbb{R})$	$\mathbb{R}X_1$	$\mathbb{R}X_2$	$\mathbb{R}X_3$	$\mathbb{R}X_4$	$\mathbb{R}X_5$	$\mathbb{R}X_6$	$\mathbb{R}X_7$	$\mathbb{R}X_8$
defined by $F_1^{(4,\mathbb{R})}$	$\mathbb{R}X_9$	$\mathbb{R}X_{10}$	$\mathbb{R}X_{11}$	$\mathbb{R}X_{12}$	$\mathbb{R}X_{13}$	$\mathbb{R}X_{14}$	$\mathbb{R}X_{15}$	
real form $su^*(4)$	$\mathbb{R}X_1$	$i\mathbb{R}X_2$	$i\mathbb{R}X_3$	$i\mathbb{R}X_4$	$\mathbb{R}X_5$	$i\mathbb{R}X_6$	$\mathbb{R}X_7$	$\mathbb{R}X_8$
defined by $F_1^{*(4)}$	$i\mathbb{R}X_9$	$\mathbb{R}X_{10}$	$i\mathbb{R}X_{11}$	$\mathbb{R}X_{12}$	$i\mathbb{R}X_{13}$	$\mathbb{R}X_{14}$	$i\mathbb{R}X_{15}$	
real form $su(2,2)$	$\mathbb{R}X_1$	$\mathbb{R}X_2$	$i\mathbb{R}X_3$	$\mathbb{R}X_4$	$\mathbb{R}X_5$	$i\mathbb{R}X_6$	$i\mathbb{R}X_7$	$i\mathbb{R}X_8$
defined by $E_3^{(2,2)}$	$i\mathbb{R}X_9$	$\mathbb{R}X_{10}$	$i\mathbb{R}X_{11}$	$\mathbb{R}X_{12}$	$i\mathbb{R}X_{13}$	$i\mathbb{R}X_{14}$	$i\mathbb{R}X_{15}$	

TABLE 21. Three real forms $L_{\mathcal{J}} = sl(4,\mathbb{R})$, $L_{\mathcal{J}} = su^*(4)$, and $L_{\mathcal{J}} = su(2,2)$ of $sl(4,\mathbb{C})$ have a fine group grading $\Gamma_6^{\mathcal{J}}$ determined by the fine group grading Γ_6 of $sl(4,\mathbb{C})$. The basis matrices of these fine group gradings are complex multiples of the matrices X_k from Table 7.

gradings Γ_2 and Γ_8 respectively, since the fourth root of unity does not belong to $\mathbb{R}$.

Thereby we show that for the real forms of the simple complex Lie algebra $sl(4,\mathbb{C})$ there exist fine group gradings which are not generated by MAD-groups of the real forms.

4.2. **Fine Group Gradings of the Real Forms of $sp(4,\mathbb{C})$ and $o(4,\mathbb{C})$.** Just like the complex Lie algebras $L = sp_K(n,\mathbb{C})$ and $L = o_K(n,\mathbb{C})$ are subalgebras of the complex Lie algebra $sl(n,\mathbb{C})$, also the real forms of L are subalgebras of the real forms of $sl(n,\mathbb{C})$. See Table 25 for the full overview of real forms of $L = sp_K(4,\mathbb{C})$ and $L = o_K(4,\mathbb{C})$ and their 'source' real forms of $sl(4,\mathbb{C})$.

Γ_7 grading subspaces	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8
	L_9	L_{10}	L_{11}	L_{12}	L_{13}	L_{14}	L_{15}	
real form $sl(4,\mathbb{R})$	$\mathbb{R}X_1$	$\mathbb{R}X_2$	$\mathbb{R}X_3$	$\mathbb{R}X_4$	$\mathbb{R}X_5$	$\mathbb{R}X_6$	$\mathbb{R}X_7$	$\mathbb{R}X_8$
defined by $F_1^{(4,\mathbb{R})}$	$\mathbb{R}X_9$	$\mathbb{R}X_{10}$	$\mathbb{R}X_{11}$	$\mathbb{R}X_{12}$	$\mathbb{R}X_{13}$	$\mathbb{R}X_{14}$	$\mathbb{R}X_{15}$	
real form $su^*(4)$	$i\mathbb{R}X_1$	$\mathbb{R}X_2$	$i\mathbb{R}X_3$	$\mathbb{R}X_4$	$i\mathbb{R}X_5$	$\mathbb{R}X_6$	$i\mathbb{R}X_7$	$\mathbb{R}X_8$
defined by $F_1^{*(4)}$	$\mathbb{R}X_9$	$i\mathbb{R}X_{10}$	$\mathbb{R}X_{11}$	$i\mathbb{R}X_{12}$	$i\mathbb{R}X_{13}$	$\mathbb{R}X_{14}$	$i\mathbb{R}X_{15}$	
real form $su(4,0)$	$i\mathbb{R}X_1$	$i\mathbb{R}X_2$	$\mathbb{R}X_3$	$\mathbb{R}X_4$	$i\mathbb{R}X_5$	$\mathbb{R}X_6$	$i\mathbb{R}X_7$	$\mathbb{R}X_8$
defined by $E_1^{(4,0)}$	$i\mathbb{R}X_9$	$\mathbb{R}X_{10}$	$\mathbb{R}X_{11}$	$i\mathbb{R}X_{12}$	$i\mathbb{R}X_{13}$	$i\mathbb{R}X_{14}$	$i\mathbb{R}X_{15}$	
real form $su(2,2)$	$\mathbb{R}X_1$	$\mathbb{R}X_2$	$i\mathbb{R}X_3$	$i\mathbb{R}X_4$	$\mathbb{R}X_5$	$i\mathbb{R}X_6$	$\mathbb{R}X_7$	$i\mathbb{R}X_8$
defined by $E_1^{(2,2)}$	$i\mathbb{R}X_9$	$\mathbb{R}X_{10}$	$\mathbb{R}X_{11}$	$i\mathbb{R}X_{12}$	$i\mathbb{R}X_{13}$	$i\mathbb{R}X_{14}$	$i\mathbb{R}X_{15}$	

TABLE 22. The fine group grading Γ_7 of $sl(4,\mathbb{C})$ determines a fine group grading $\Gamma_7^{\mathcal{J}}$ for four of the real forms of $sl(4,\mathbb{C})$, namely $L_{\mathcal{J}} = sl(4,\mathbb{R})$, $L_{\mathcal{J}} = su^*(4)$, $L_{\mathcal{J}} = su(4,0)$, and $L_{\mathcal{J}} = su(2,2)$. Their bases are given by means of the matrices X_k from Table 8.

Γ_8 grading subspaces	L_1	L_2	L_3	L_4	L_5
	L_6	L_7	L_8	L_9	L_{10}
	L_{11}	L_{12}	L_{13}	L_{14}	
real form $sl(4,\mathbb{R})$	$\mathbb{R}X_1$	$\mathbb{R}X_2$	$\mathbb{R}X_3$	$\mathbb{R}X_4$	$\mathbb{R}X_5$
defined by $F_1^{(4,\mathbb{R})}$	$\mathbb{R}X_6$	$\mathbb{R}X_7$	$\mathbb{R}X_8$	$\mathbb{R}X_9$	$\mathbb{R}X_{10}$
	$\mathbb{R}X_{11}$	$\mathbb{R}X_{12}$	$\mathbb{R}X_{13}$	$\mathbb{R}X_{14} + \mathbb{R}X_{15}$	
real form $su^*(4)$	$(1-i)\mathbb{R}X_1$	$(1+i)\mathbb{R}X_2$	$(1+i)\mathbb{R}X_3$	$(1-i)\mathbb{R}X_4$	$(1+i)\mathbb{R}X_5$
defined by $F_2^{*(4)}$	$(1-i)\mathbb{R}X_6$	$(1-i)\mathbb{R}X_7$	$(1+i)\mathbb{R}X_8$	$\mathbb{R}X_9$	$i\mathbb{R}X_{10}$
	$i\mathbb{R}X_{11}$	$\mathbb{R}X_{12}$	$\mathbb{R}X_{13}$	$i\mathbb{R}X_{14} + i\mathbb{R}X_{15}$	
real form $su(3,1)$	$i\mathbb{R}X_1$	$i\mathbb{R}X_2$	$\mathbb{R}X_3$	$\mathbb{R}X_4$	$i\mathbb{R}X_5$
defined by $E_2^{(3,1)}$	$\mathbb{R}X_6$	$\mathbb{R}X_7$	$\mathbb{R}X_8$	$i\mathbb{R}X_9$	$i\mathbb{R}X_{10}$
	$\mathbb{R}X_{11}$	$\mathbb{R}X_{12}$	$i\mathbb{R}X_{13}$	$i\mathbb{R}X_{14} + i\mathbb{R}X_{15}$	
real form $su(2,2)$	$(1+i)\mathbb{R}X_1$	$(1+i)\mathbb{R}X_2$	$(1-i)\mathbb{R}X_3$	$(1-i)\mathbb{R}X_4$	$(1-i)\mathbb{R}X_5$
defined by $E_5^{(2,2)}$	$(1+i)\mathbb{R}X_6$	$(1-i)\mathbb{R}X_7$	$(1+i)\mathbb{R}X_8$	$i\mathbb{R}X_9$	$i\mathbb{R}X_{10}$
	$\mathbb{R}X_{11}$	$i\mathbb{R}X_{12}$	$\mathbb{R}X_{13}$	$\mathbb{R}X_{14} + \mathbb{R}X_{15}$	

TABLE 23. Also the fine group grading Γ_8 of $sl(4,\mathbb{C})$ determines a fine group grading $\Gamma_8^{\mathcal{J}}$ for four of the real forms of $sl(4,\mathbb{C})$, this time for $L_{\mathcal{J}} = sl(4,\mathbb{R})$, $L_{\mathcal{J}} = su^*(4)$, $L_{\mathcal{J}} = su(3,1)$, and $L_{\mathcal{J}} = su(2,2)$. Their bases derive from the matrices X_k given in Table 9 via multiplication by complex coefficients listed here.

Having this direct link between the real forms of $sl(4,\mathbb{C})$ and those of its subalgebras $L \subset sl(4,\mathbb{C})$, we easily obtain fine gradings of the real forms of L, because they are displayed by the fine gradings of the respective 'source' real forms of $sl(4,\mathbb{C})$.

The choice of the representation matrix K is given already by the way the complex subalgebra is displayed by the fine gradings of the complex $sl(4,\mathbb{C})$ - see $K_0 = J$ for $sp(4,\mathbb{C})$ in Table 10 and K_1, K_2, K_3 for $o(4,\mathbb{C})$ in Table 11.

4.2.1. *Fine Group Gradings of the Real Forms of* $sp(4,\mathbb{C})$. When describing the fine group gradings of the real forms of $sp(4,\mathbb{C})$, we stay in just one representation $sp_{K_0}(4,\mathbb{C}) = sp_J(4,\mathbb{C})$ - as corresponds to the complex case. As the complex subalgebra $sp(4,\mathbb{C})$ of $sl(4,\mathbb{C})$ is displayed just by three fine group gradings of $sl(4,\mathbb{C})$ - namely Γ_5, Γ_6, Γ_7, it is only these three cases that come into play also for the real forms:

- The real form $sp(4,\mathbb{R})$ has fine group gradings determined by the original fine group gradings Γ_5, Γ_6, Γ_7 of $sl(4,\mathbb{C})$:
  $\Gamma_5, \Gamma_6, \Gamma_7 \quad \rightarrow \quad sp(4,\mathbb{R}) = sp_J(4,\mathbb{C}) \cap sl_{F_1}(4,\mathbb{R})$

real form of $sl(4,\mathbb{C})$	fine gradings obtained by 'MAD-group' method	fine gradings obtained by 'real basis' or 'fundamental' method only
$sl(4,\mathbb{R})$	$\Gamma_1^{\mathcal{J}},\Gamma_3^{\mathcal{J}},\Gamma_4^{\mathcal{J}},\Gamma_5^{\mathcal{J}},\Gamma_6^{\mathcal{J}},\Gamma_7^{\mathcal{J}}$	$\Gamma_8^{\mathcal{J}}$
$su^*(4)$	$\Gamma_6^{\mathcal{J}},\Gamma_7^{\mathcal{J}}$	$\Gamma_8^{\mathcal{J}}$
$su(4,0)$	$\Gamma_3^{\mathcal{J}},\Gamma_7^{\mathcal{J}}$	
$su(3,1)$	$\Gamma_3^{\mathcal{J}},\Gamma_4^{\mathcal{J}}$	$\Gamma_2^{\mathcal{J}},\Gamma_8^{\mathcal{J}}$
$su(2,2)$	$\Gamma_3^{\mathcal{J}},\Gamma_4^{\mathcal{J}},\Gamma_5^{\mathcal{J}},\Gamma_6^{\mathcal{J}},\Gamma_7^{\mathcal{J}}$	$\Gamma_2^{\mathcal{J}},\Gamma_8^{\mathcal{J}}$

TABLE 24. Summary of the fine group gradings of real forms of $sl(4,\mathbb{C})$: Those determined by Γ_1, Γ_3, Γ_4, Γ_5, Γ_6, Γ_7 can be obtained as decompositions of the real forms into simultaneous eigensubspaces of automorphisms from MAD-groups of the real forms, which is not the case of the fine group gradings determined by Γ_2 and Γ_8.

real form of $sl(4,\mathbb{C})$	real form of $sp(4,\mathbb{C})$	real form of $o(4,\mathbb{C})$
$sl(4,\mathbb{R})$	$sp(4,\mathbb{R})$	
$su^*(4)$		$so^*(4)$
$su(4,0)$	$usp(4,0)$	$so(4,0)$
$su(3,1)$		$so(3,1)$
$su(2,2)$	$usp(2,2)$	$so(2,2)$

TABLE 25. The full list of the real forms of $sp(4,\mathbb{C})$ and of $o(4,\mathbb{C})$, and their relation to the real forms of $sl(4,\mathbb{C})$.

- The real form $usp(4,0)$ has a fine group grading determined by Γ_7:
  $\Gamma_7 \quad \rightarrow \quad usp(4,0) = sp_J(4,\mathbb{C}) \cap su_{E_1}(4,0)$
- The real form $usp(2,2)$ has fine group gradings determined by Γ_5, Γ_6, Γ_7:
  $\Gamma_5,\Gamma_6 \quad \rightarrow \quad usp(2,2) = sp_J(4,\mathbb{C}) \cap su_{E_3}(2,2)$
  $\Gamma_7 \quad \rightarrow \quad usp(2,2) = sp_J(4,\mathbb{C}) \cap su_{E_1}(2,2)$

real form of $sp(4,\mathbb{C})$	fine gradings displayed by fine gradings of real forms of $sl(4,\mathbb{C})$
$sp(4,\mathbb{R})$	$\Gamma_5^{\mathcal{J}},\Gamma_6^{\mathcal{J}},\Gamma_7^{\mathcal{J}}$
$usp(4,0)$	$\Gamma_7^{\mathcal{J}}$
$usp(2,2)$	$\Gamma_5^{\mathcal{J}},\Gamma_6^{\mathcal{J}},\Gamma_7^{\mathcal{J}}$

TABLE 26. Summary of fine group gradings of the real forms of $sp(4,\mathbb{C})$ displayed by the fine group gradings of the real forms of $sl(4,\mathbb{C})$.

4.2.2. *Fine Group Gradings of the Real Forms of* $o(4,\mathbb{C})$. Here we must work with more representations of the algebra $o(4,\mathbb{C})$ already - like in the complex case. The different representations that come into play are those defined by symmetric matrices K_1, K_2, and K_3 listed in Table 11. The complex subalgebra $o(4,\mathbb{C})$ is displayed by six fine group gradings of $sl(4,\mathbb{C})$ - namely Γ_3, Γ_4, Γ_5, Γ_6, Γ_7, Γ_8. Thus just these six group gradings display fine group gradings for real forms of $o(4,\mathbb{C})$, in those cases that are relevant for the respective real form.

- The real form $so^*(4)$ has fine group gradings determined by the original fine group gradings Γ_6, Γ_7, Γ_8 of $sl(4,\mathbb{C})$:
  $\Gamma_6 \quad \rightarrow \quad so^*(4) = o_{K_3}(4,\mathbb{C}) \cap su^*_{F_1}(4)$
  $\Gamma_7 \quad \rightarrow \quad so^*(4) = o_{K_1}(4,\mathbb{C}) \cap su^*_{F_1}(4)$
  $\Gamma_8 \quad \rightarrow \quad so^*(4) = o_{K_2}(4,\mathbb{C}) \cap su^*_{F_2}(4)$
- The real form $so(4,0)$ has fine group gradings determined by Γ_3 and Γ_7:
  $\Gamma_3,\Gamma_7 \quad \rightarrow \quad so(4,0) = o_{K_1}(4,\mathbb{C}) \cap su_{E_1}(4,0)$

- The real form $so(3,1)$ has fine group gradings determined by Γ_3, Γ_4, Γ_8:
  $\Gamma_3 \quad \rightarrow \quad so(3,1) = o_{K_1}(4,\mathbb{C}) \cap su_{E_1}(3,1)$
  $\Gamma_4, \Gamma_8 \quad \rightarrow \quad so(3,1) = o_{K_2}(4,\mathbb{C}) \cap su_{E_2}(3,1)$
- The real form $so(2,2)$ has fine group gradings determined by Γ_3, Γ_4, Γ_5, Γ_6, Γ_7, and Γ_8:
  $\Gamma_3, \Gamma_7 \quad \rightarrow \quad so(2,2) = o_{K_1}(4,\mathbb{C}) \cap su_{E_1}(2,2)$
  $\Gamma_4 \quad \rightarrow \quad so(2,2) = o_{K_2}(4,\mathbb{C}) \cap su_{E_2}(2,2)$
  $\Gamma_5, \Gamma_6 \quad \rightarrow \quad so(2,2) = o_{K_3}(4,\mathbb{C}) \cap su_{E_3}(2,2)$
  $\Gamma_8 \quad \rightarrow \quad so(2,2) = o_{K_2}(4,\mathbb{C}) \cap su_{E_5}(2,2)$

real form of $o(4,\mathbb{C})$	fine gradings displayed by fine gradings of real forms of $sl(4,\mathbb{C})$
$so^*(4)$	$\Gamma_6^{\mathcal{J}}, \Gamma_7^{\mathcal{J}}, \Gamma_8^{\mathcal{J}}$
$so(4,0)$	$\Gamma_3^{\mathcal{J}}, \Gamma_7^{\mathcal{J}}$
$so(3,1)$	$\Gamma_3^{\mathcal{J}}, \Gamma_4^{\mathcal{J}}, \Gamma_8^{\mathcal{J}}$
$so(2,2)$	$\Gamma_3^{\mathcal{J}}, \Gamma_4^{\mathcal{J}}, \Gamma_5^{\mathcal{J}}, \Gamma_6^{\mathcal{J}}, \Gamma_7^{\mathcal{J}}, \Gamma_8^{\mathcal{J}}$

TABLE 27. Summary of fine group gradings of the real forms of $o(4,\mathbb{C})$ displayed by the fine group gradings of the real forms of $sl(4,\mathbb{C})$.

5. Concluding remarks

In the whole article, we deal only with fine group gradings of real Lie algebras, but we do not study non-group gradings. Whereas on complex Lie algebras there is a one-to-one correspondence between fine group gradings and MAD-groups, on real forms of $sl(4,\mathbb{C})$ we have found fine group gradings which are not generated by any MAD-group on the respective real form. The question whether our list of fine group gradings of these real forms is complete still remains open. This problem requires further investigation, similarly as the question of existence of non-group fine gradings.

Even for the known gradings, a number of additional questions can be raised. Let us point out a few:

- The gradings of Lie algebras should be extended to their representations, finite as well as infinite dimensional ones.
- Lie algebras often need to be extended (semidirect product) by an Abelian algebra (translations). Best known case is the Poincaré Lie algebra of space-time transformations extending $o(3,1)$. Fine gradings of such algebras would be of interest to know.
- Fine gradings which decompose the Lie algebra into one-dimensional spaces define a basis of the algebra reflecting its unique structure, certainly a particular simplicity of corresponding structure constants.
- Gradings of an algebra L frequently display particular subalgebras of L. That is, such subalgebras are formed by one or several grading subspaces. All of the fine gradings of L together display many subalgebras. Which maximal subalgebras are displayed, and which are missing?
- Casimir operators should have rather different form in different fine gradings. Casimir operators of displayed subalgebras should be clearly recognized there.

Acknowledgements

We are grateful to the referee, who drew our attention to the problems connected with the difference between the terms 'grading' and 'group grading'. His / her comments have significantly improved the quality of our text. The referee has, among other, pointed out that the Theorem 6 can be stated as equivalence, and has provided a proof of the implication that was originally missing.

The authors acknowledge financial support of the Doppler Institute by the grant LC06002 of the Ministry of Education, Youth and Sports of the Czech Republic.

References

[1] C. Bahri, D.J. Rowe, and W. Wijesundera, *Phase transition in the pairing-plus-quadrupole model*, Phys. Rev. C **58** (1998) 1539-1550

[2] Y.A. Bahturin, I.P. Shestakov, M.V. Zaicev, *Gradings on simple Jordan and Lie algebras*, J.Algebra **283** (2005) 849-868

[3] S.D. Bartlett, D.A. Rice, B.C. Sanders, H. de Guise, *Unitary transformations for testing Bell inequalities,* Phys. Rev. A. **63** 10 pp.

[4] C. Draper, C. Martín, *Gradings on g_2,* Linear Algebra and its Applications **418** (2006) 85-111

[5] C. Draper, C. Martín, *Gradings on the Albert algebra and on f_4,* arXiv: math/0703840

[6] A. Elduque, *A Lie grading which is not a semigroup grading,* Linear Algebra and its Applications **418** (2006) 312-314

[7] M. Havlíček, J. Patera, and E. Pelantová, *On Lie gradings II* (dedicated to the memory of H. Zassenhaus), Linear Algebra and its Applic. **277** (1998) 97–125

[8] M. Havlíček, J. Patera, and E. Pelantová, *On Lie gradings III. Gradings of the real forms of classical Lie algebras,* (dedicated to the memory of H. Zassenhaus), Linear Algebra and its Applic. **314** (2000) 1–47

[9] N. Jacobson, *Lie Algebras,* Dover, New York, 1979

[10] A.U. Klimyk, *The nonstandard q-deformation of the enveloping algebra $U(so_n)$: results and problems,* Czech. J. Phys. **51** 331 (2001)

[11] A.U. Klimyk and K. Schmüdgen, *Quantum groups and their representations,* Springer 1997

[12] X. Leng and J. Patera, *Graded contractions of representations of orthogonal and symplectic Lie algebras with respect to their maximal parabolic subalgebras,* J. Phys. A: Math. Gen., **28** (1995) 3785-3807

[13] M. de Montigny, J. Patera, *Discrete and continuous graded contractions of Lie algebras and superalgebras,* J. Phys. A: Math. Gen., **24** (1991) 525-547

[14] J. Patera, E. Pelantová, M. Svobodová, *The eight fine gradings of $sl(4,C)$ and $o(6,C)$,* J. Math. Phys. **43** (2002) 6353–6378

[15] J. Patera, E. Pelantová, M. Svobodová, *Fine gradings of $o(5,C)$, $sp(4,C)$ and of their real forms,* J. Math. Phys. **42** (2001) 3839–3853

[16] J. Patera, E. Pelantová, M. Svobodová, *Fine gradings of $o(4,C)$,* J. Math. Phys. **45** (2004) 2188–2198

[17] J. Patera, H. Zassenhaus, *On Lie gradings I,* Linear Algebra and its Applications **112** (1989) 87-159

[18] D.J. Rowe, *Microscopic theory of nuclear collective motion,* Rep. Progr. Phys. **48** (1985) 1419-148

[19] D.J. Rowe, *Dynamical symmetries of nuclear collective models,* Prog. Part. Nucl. Phys. **37** (1996) 265-348

[20] P. Šťovíček, J. Tolar, *Quantum Mechanics in a Discrete Space-time,* Rep. Math. Phys. **20** (1984) 157-170

[21] H. Weyl, *Theory of Groups and Quantum Mechanics,* Dover, New York 1950, 272-280

Centre de recherches mathématiques, Université de Montréal, C.P.6128-Centre ville, Montréal, H3C 3J7, Québec, Canada

E-mail address: patera@crm.umontreal.ca

Doppler Institute and Department of mathematics, Faculty of Nuclear Science and Physical Engineering, Czech Technical University, Trojanova 13, Praha 2, 120 00, Czech Republic

E-mail address: pelantova@km1.fjfi.cvut.cz

Doppler Institute and Department of mathematics, Faculty of Nuclear Science and Physical Engineering, Czech Technical University, Trojanova 13, Praha 2, 120 00, Czech Republic

E-mail address: svobodov@km1.fjfi.cvut.cz